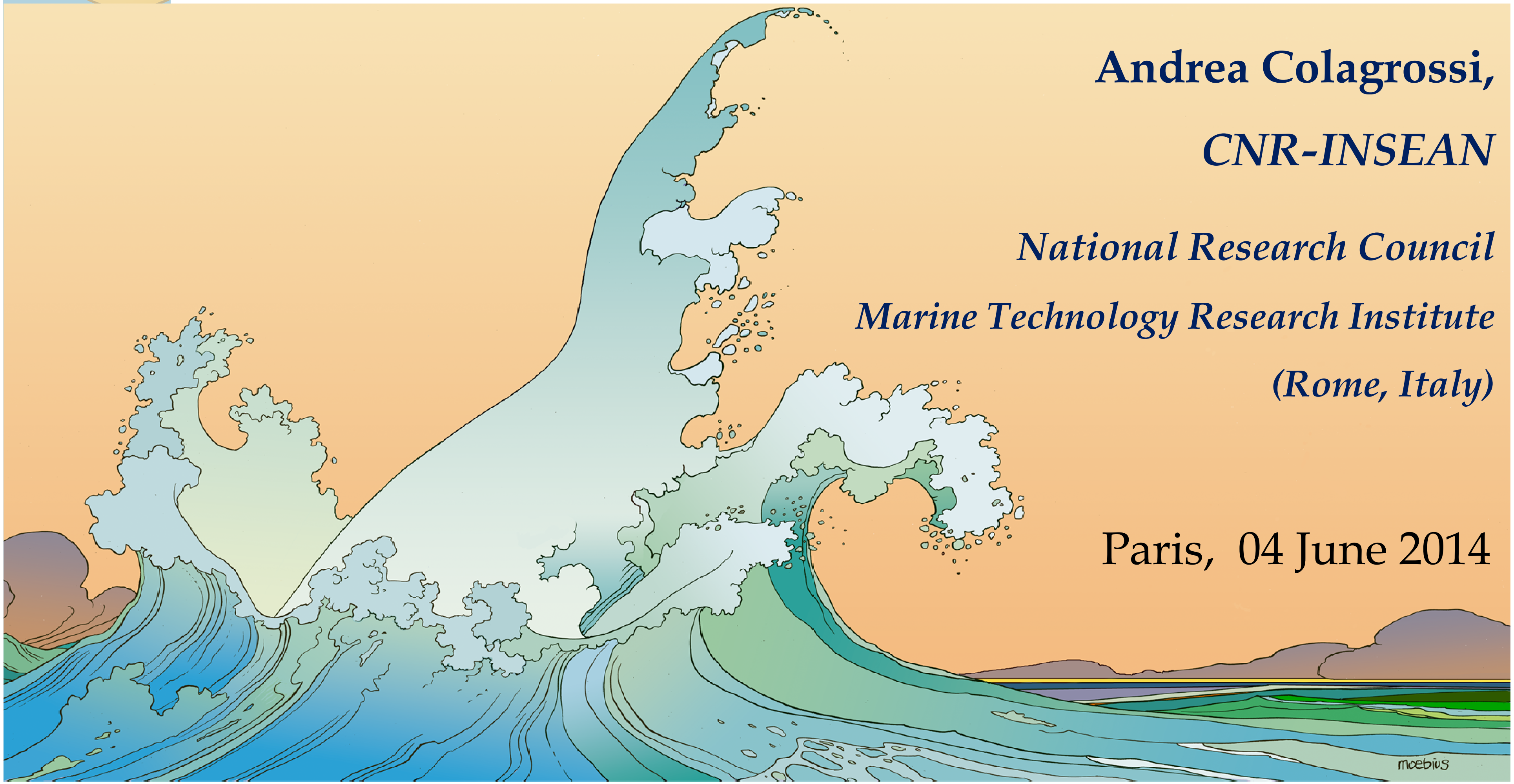
Simulating free-surface viscous flows with SPH: Theoretical and Practical aspects.



Andrea Colagrossi,

CNR-INSEAN

National Research Council

Marine Technology Research Institute

(Rome, Italy)

Paris, 04 June 2014

Framework and Aims of the Lecture

• **Research Activity (1999-2014)**:

- Development of SPH schemes aimed at simulating 3D Free-surface and Interface Flows with large deformations, including Breaking and Fragmentation of the Interface.

- Theoretical analysis of the SPH schemes.

- Collect some useful practical information for simulating violent free-surface flows.

Research Activity Teams from 1999-2014

- Maurizio Landrini **(INSEAN)**
- Marshall Peter Tulin **(OEL , UCSB, S. Barbara, CA)**
- Marilena Greco **(University of Trondheim, Norway)**
- A. Souto-Iglesias, Louis Delorme **(Technical University of Madrid)**
- David Le Touzé , Nicolas Grenier **(Ecole Centrale de Nantes)**
- Diego Molteni **(University of Palermo)**
- **Matteo Antuono, Salvatore Marrone** **(CNR-INSEAN)**
- Giorgio Graziani **(University of Rome "Sapienza")**
- Claudio Lugni, Giuseppina Colicchio **(CNR-INSEAN)**
- F. Masia, L. Gonzales **(Technical University of Madrid)**
- Joseph J. Monaghan **(University of Monash)**
- Ivan Federico **(University of Calabria, Arcavacata di Rende (CS), Italy)**
- Benjamin Bouscasse **(CNR-INSEAN)**
- Mario Pulvirenti, Emanuele Rossi **(University of Rome "Sapienza")**
- Andrea Di Mascio **(CRN – IAC)** **… …**

October 1999

"Simulation of ship breaking waves"

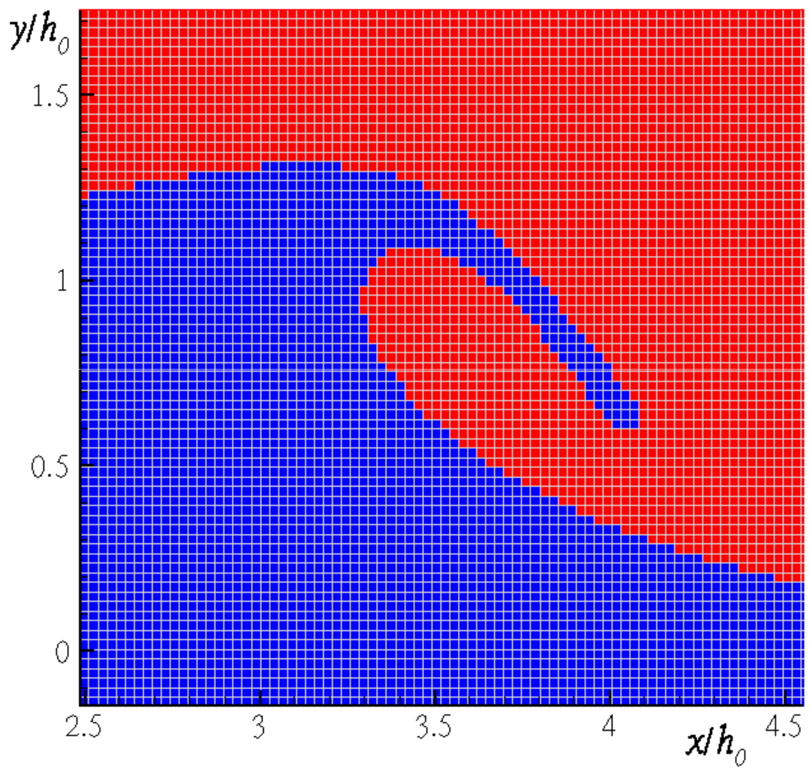
Surface Ship Hydrodynamics

Program of the Office of Naval Research

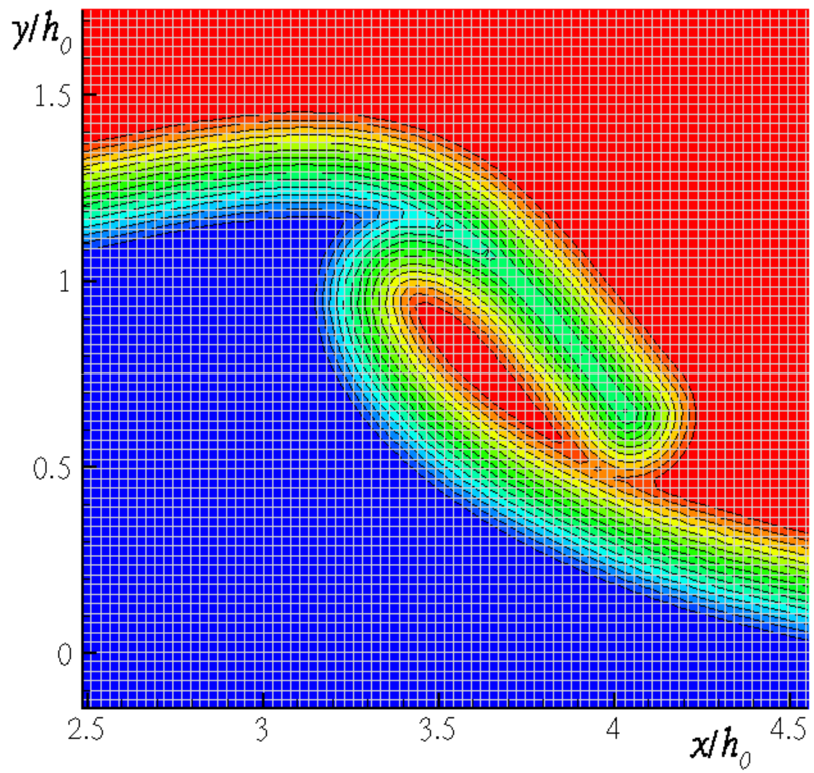
Prof. M.P. Tulin, Director of Ocean Engineering Laboratory, UCSB, S. Barbara, CA

Wave Breaking phenomena

Eulerian Mesh based solver



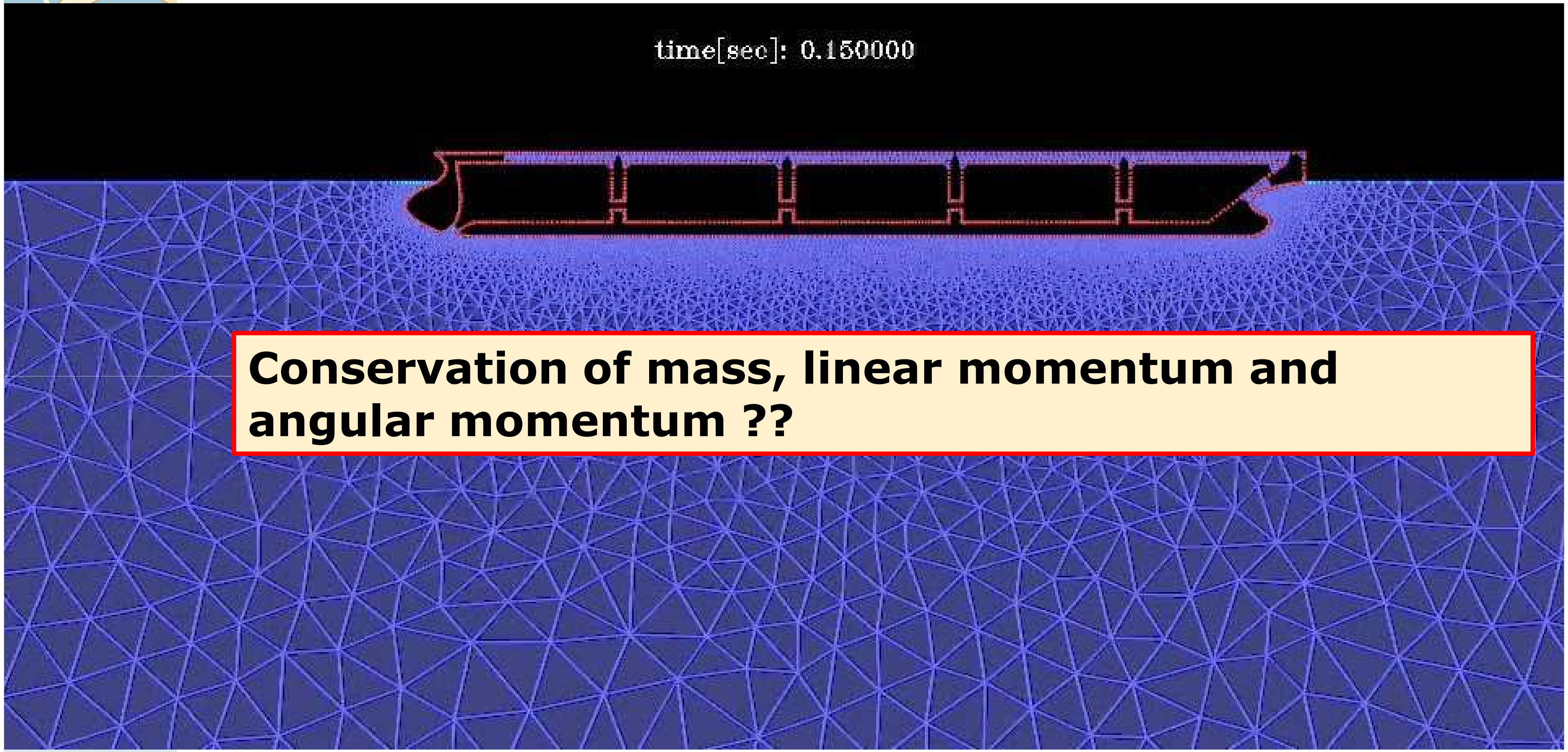
Volume Of Fluid (VOF)
C.W. Hirt & B.D. Nichols (1981)



Level Set Method
J.A. Sethian (1999)

Particles + Mesh (P-FEM)

Damaged ship simulation (2001)



Conservation of mass, linear momentum and angular momentum ??

Idelsohn SR, Storti MA, Oñate E. *Lagrangian formulations to solve free surface incompressible inviscid fluid flows.* Computer Methods in Applied Mechanics and Engineering 2001; 191:583–593.

SPH for Free-Surface flow

J.J. Monaghan 1994

JOURNAL OF COMPUTATIONAL PHYSICS 110, 399–406 (1994)

Simulating Free Surface Flows with SPH

J. J. MONAGHAN

Department of Mathematics, Monash University, Clayton Victoria 3168, Australia

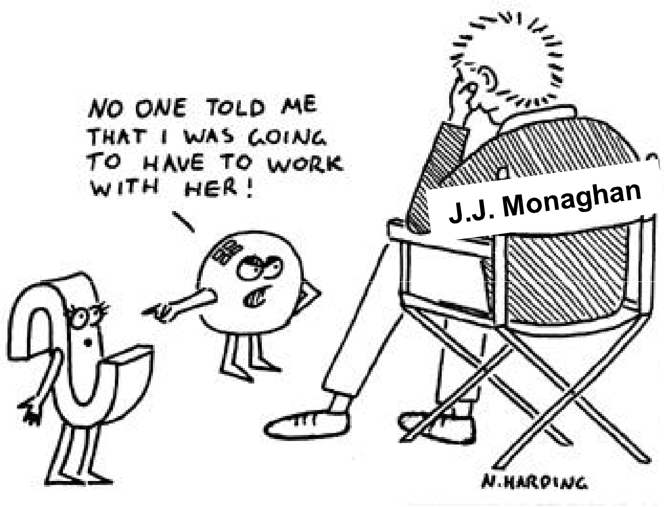
Received October 16, 1992

The SPH (smoothed particle hydrodynamics) method is extended to deal with free surface incompressible flows. The method is easy to use, and examples will be given of its application to a breaking dam, a bore, the simulation of a wave maker, and the propagation of waves towards a beach. Arbitrary moving boundaries can be included by modelling the boundaries by particles which repel the fluid particles. The method is explicit, and the time steps are therefore much shorter than required by other less flexible methods, but it is robust and easy to program.

1. INTRODUCTION

these is to work directly with the constraint of constant density. It is possible to include these constraints easily in the SPH formalism by using the Gibbs–Appell equations [15] which are generalized versions of Gauss' principle of least constraint. Unfortunately, the resulting equations are cumbersome, and it has not been possible to solve them efficiently without further approximations.

The second approach, and the one we use here, is based on the observation that real fluids such as water are compressible, but with a speed of sound which is very much greater than the speed of bulk flow. The momentum equation shows that the variation in density $\delta\rho$ is given by



1328 citations (one of the most cited SPH-article)

Damaged ship simulation

MPS (Naito, S., Sueyoshi, M., 2002)

Simulations of free-surface flows using SPH/MPS methods exhibit a higher realism, for these reasons they were also largley applied in the context of computer-graphics. Further the algorithms are generally simpler than Mesh-based methods.

VICTOR GONZÁLEZ, (C.E.O. OF THE "NEXT LIMIT" Ltd.) TECHINCAL OSCAR IN 2008 FOR THEIR WORK IN

LORD OF THE RINGS.

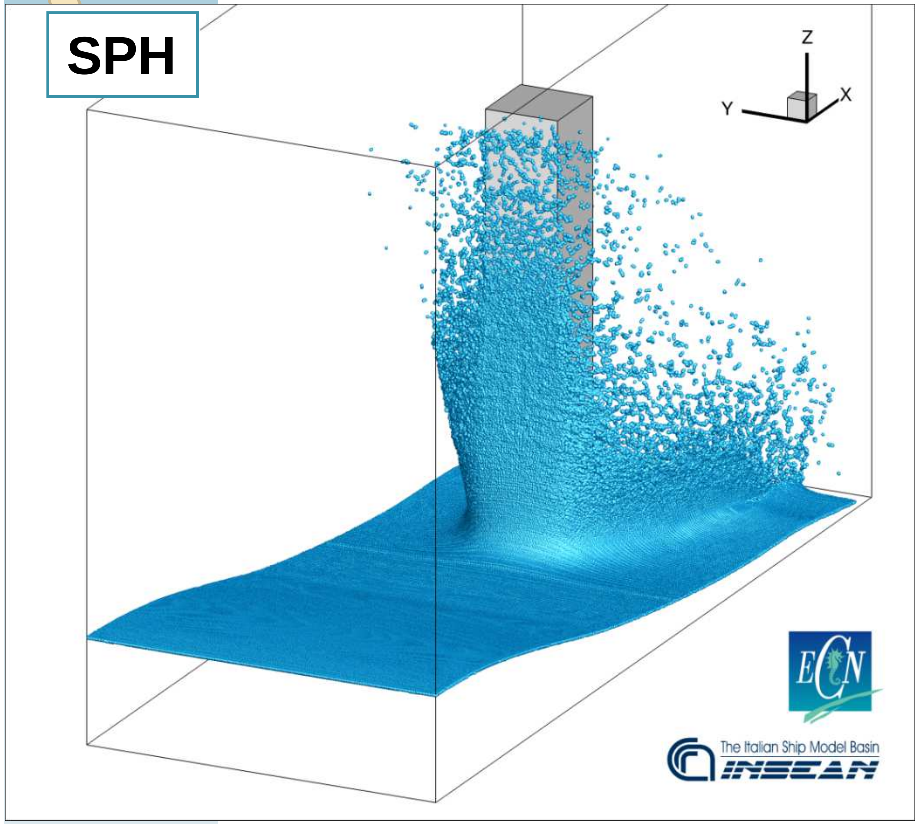
Classification of Numerical Methods able to deal with Violent Free-Surface problems.

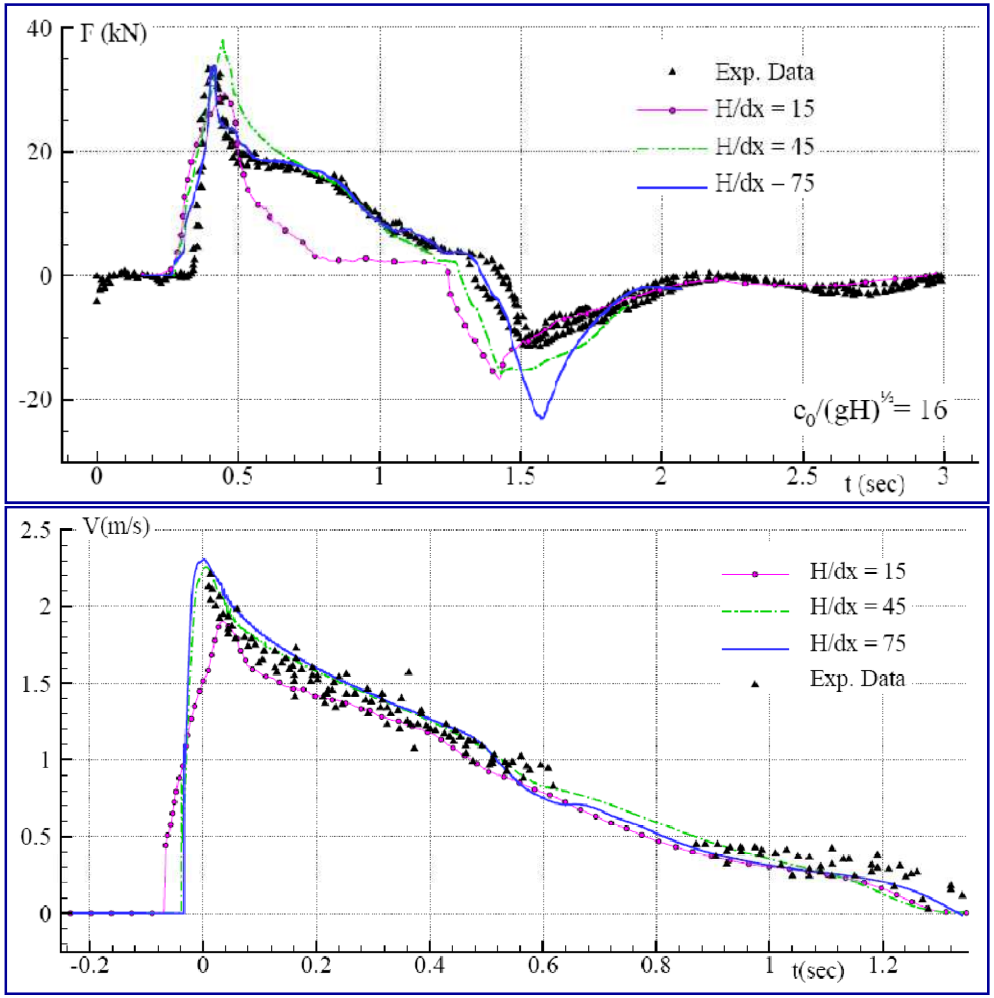
	Eulerian (Interface Capturing/tracking) (Level Set, VOF, MAC, CIP)	Lagrangian
Grid Based	FD Method, FEM, FVM	Deformable Mesh, P-FEM
Meshless	FEM based on Integral Interpolation	Particle Methods (SPH, MPS …)

SPH – ***S**moothed **P**article **H**ydrodynamics*

Smoothed Particle Hydrodynamics Method

Violent water-impact flows (2006)

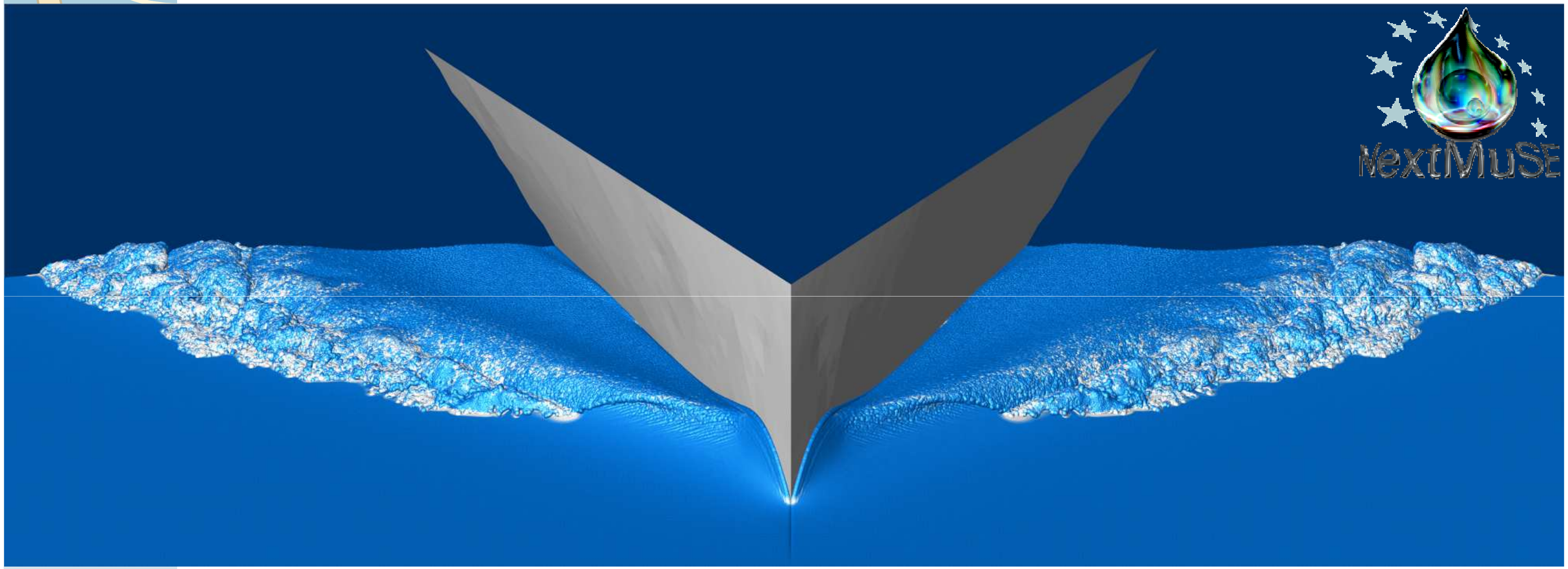




First Benchmark test-case
Le Touzé et al. 1st SPHERIC Workshop, Rome (2006)

Breaking wave pattern generated by ships

3D SPH model

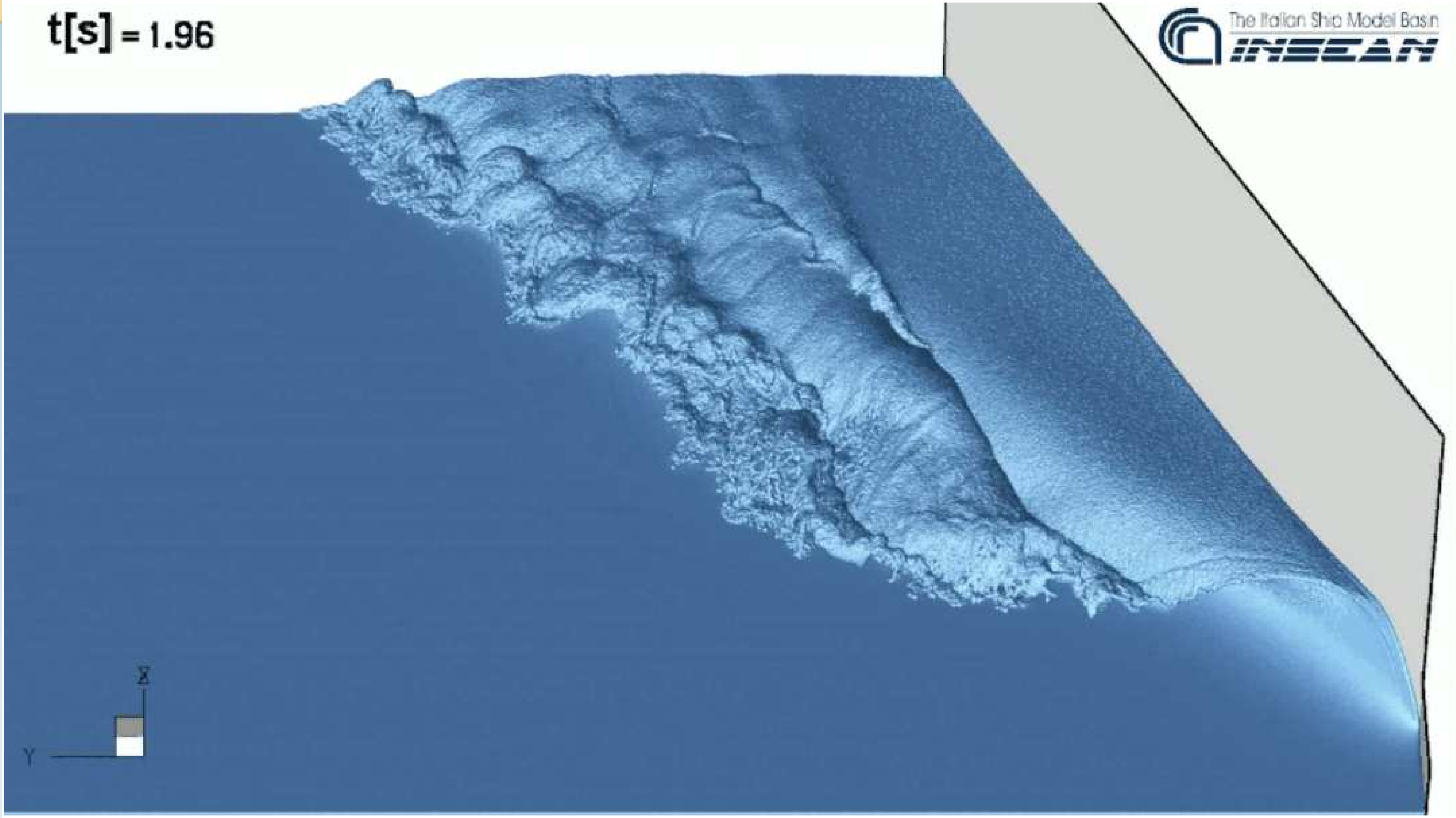


Parallel SPH simulations on Cluster machine up to 10^8 particles

S. Marrone, B. Bouscasse , A. Colagrossi, M. Antuono**, Study of ship wave breaking patterns using 3D parallel SPH simulations**, *Computers & Fluids*, 69, 54–66, (2012)

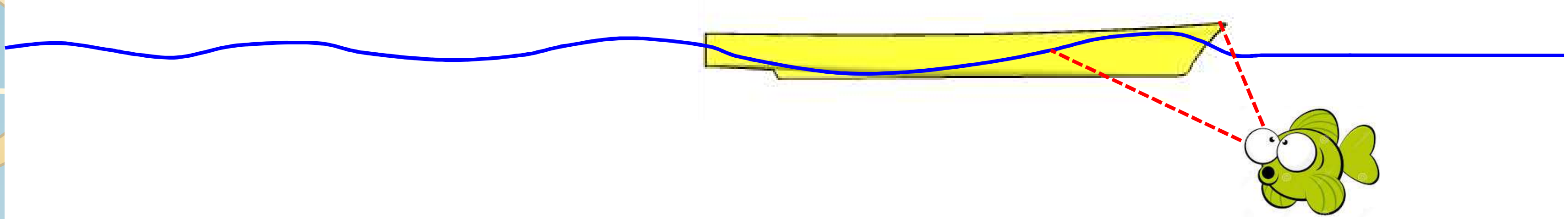
Breaking wave pattern generated by ships

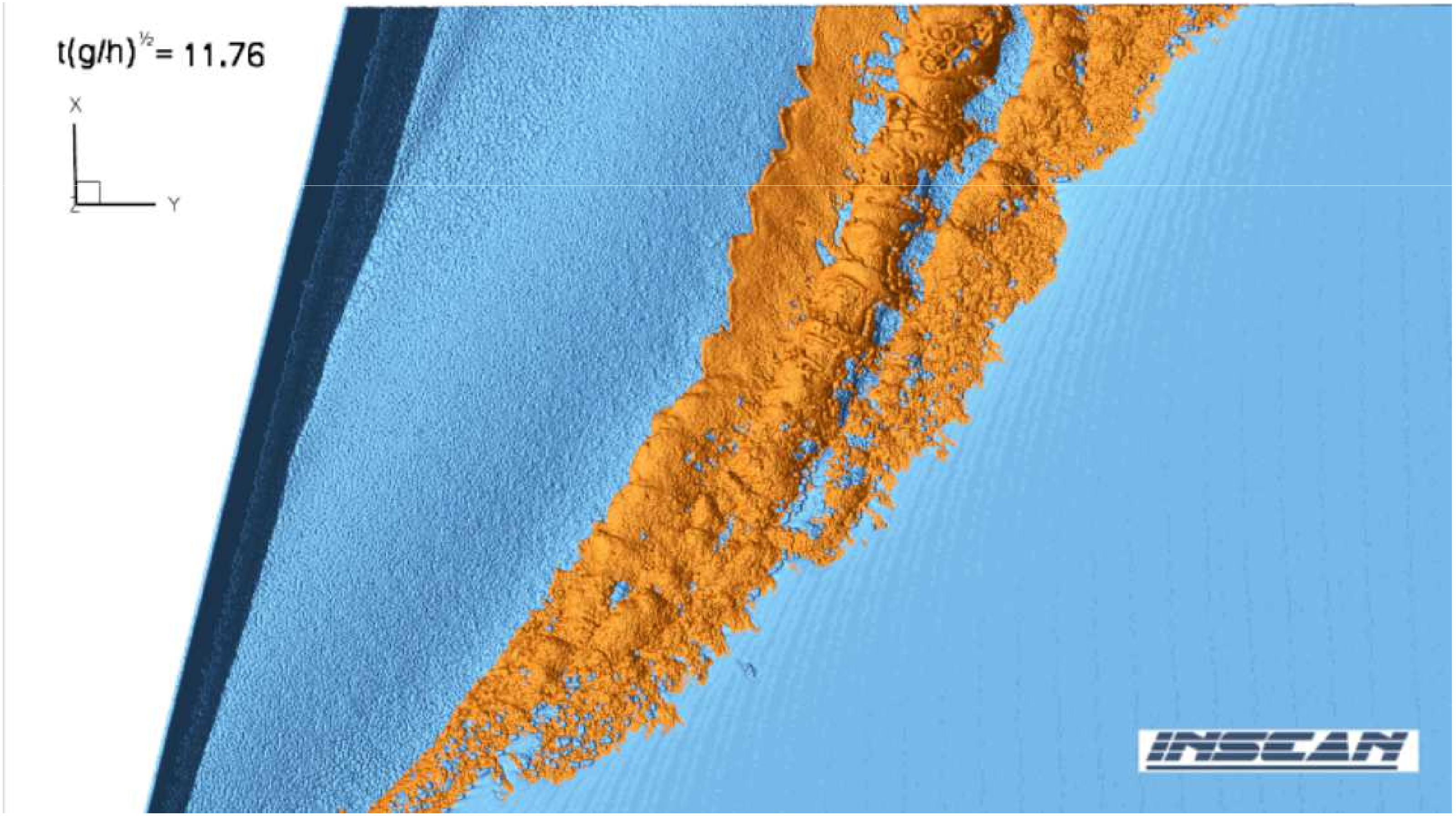
3D SPH model



Breaking wave pattern generated by ships

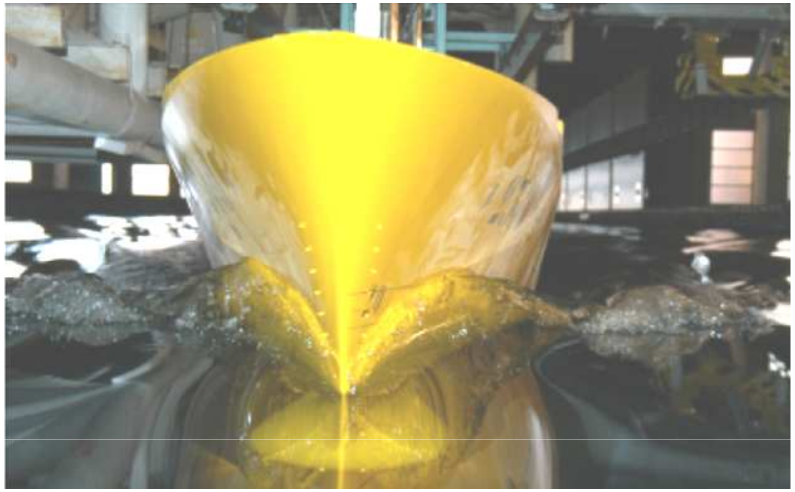
3D SPH model

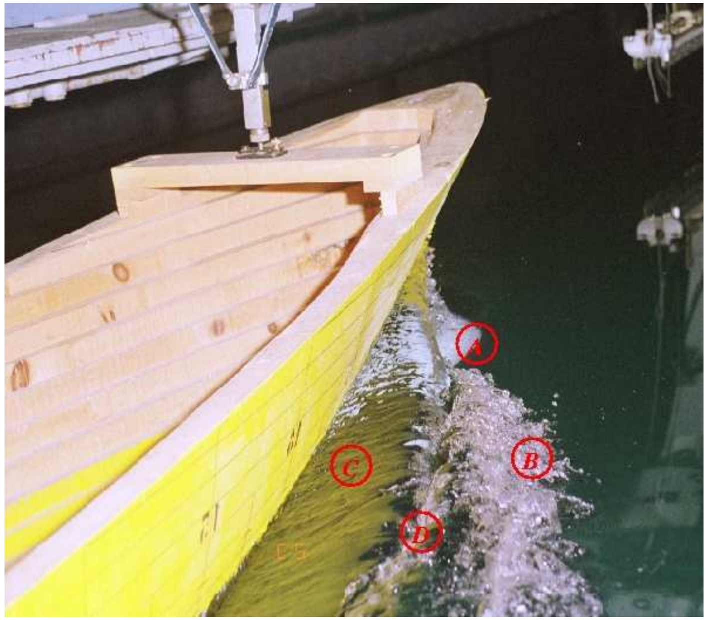


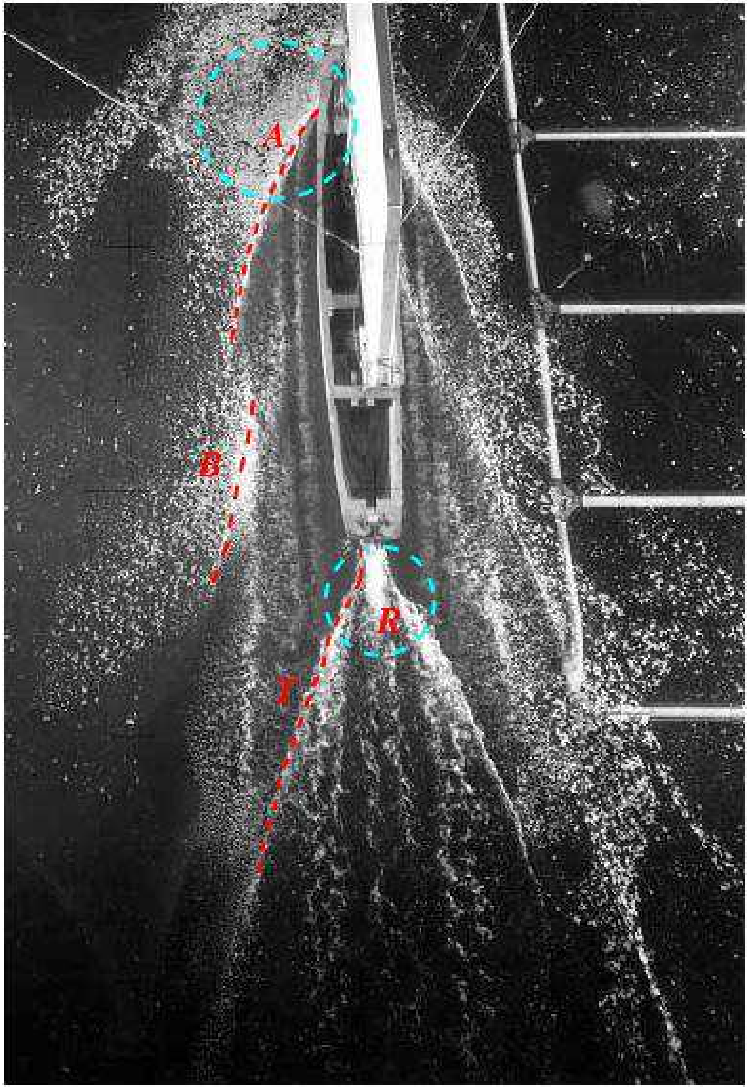


Breaking wave pattern generated by ships

DTMB model of DDG51, Fr = 0.41)

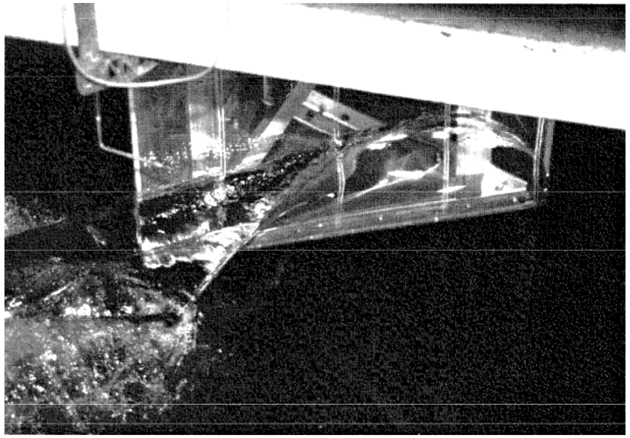


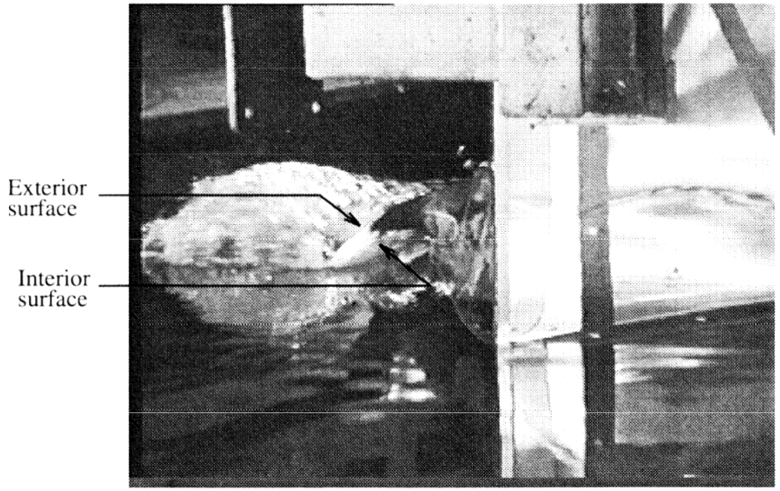


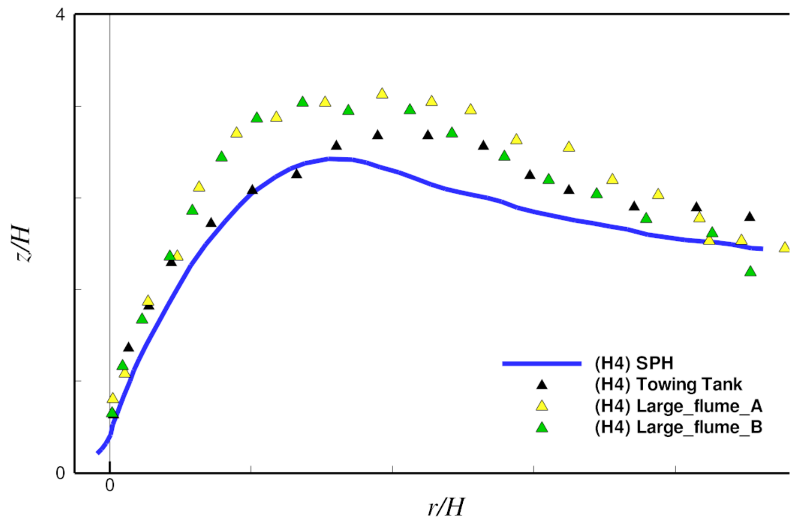


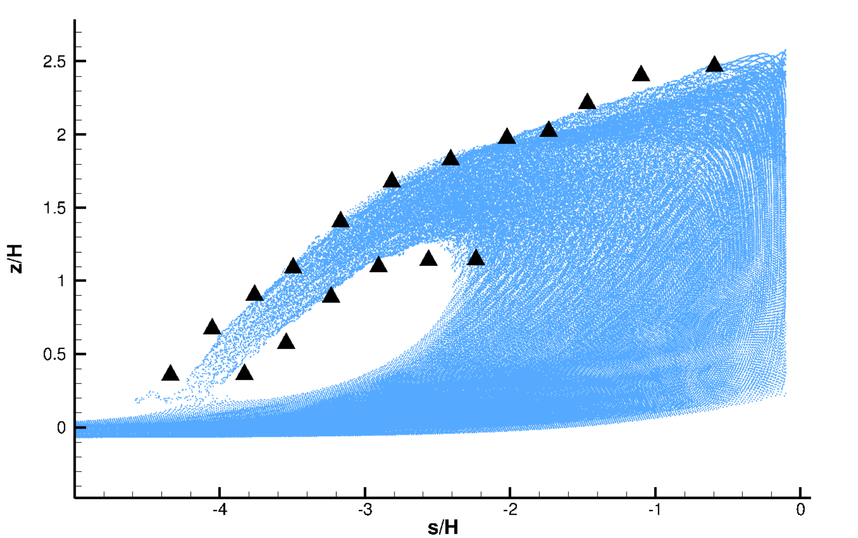
Breaking wave pattern generated by ships

3D SPH model









Smoothed Particle Hydrodynamics Method

Theoretical Analysis for free-surface and interfacial flows

1. Colagrossi, M. Landrini, *Numerical Simulation of Interfacial Flows by Smoothed* Particle Hydrodynamics, Journal of Computational Physics, 191, N.2, p. 448-475, 2003.

2. N. Grenier, M. Antuono, A. Colagrossi, D. Le Touzé, B. Alessandrini, *An Hamiltonian interface SPH formulation for multi-fluid and free surface flows*, Journal of Computational Physics 228, 8380–8393, 2009.

3. Colagrossi, M. Antuono, D. Le Touzé, *Theoretical considerations on the free-surface role in the smoothed-particle-hydrodynamics model*, Physical Review E, 79, 1-13, 2009.

4. M. Antuono, A. Colagrossi, S. Marrone, D. Molteni, *Free-surface flows solved by means of SPH schemes with numerical diffusive terms*, Computer Physics Communications, 181(3): 532-549, 2010.

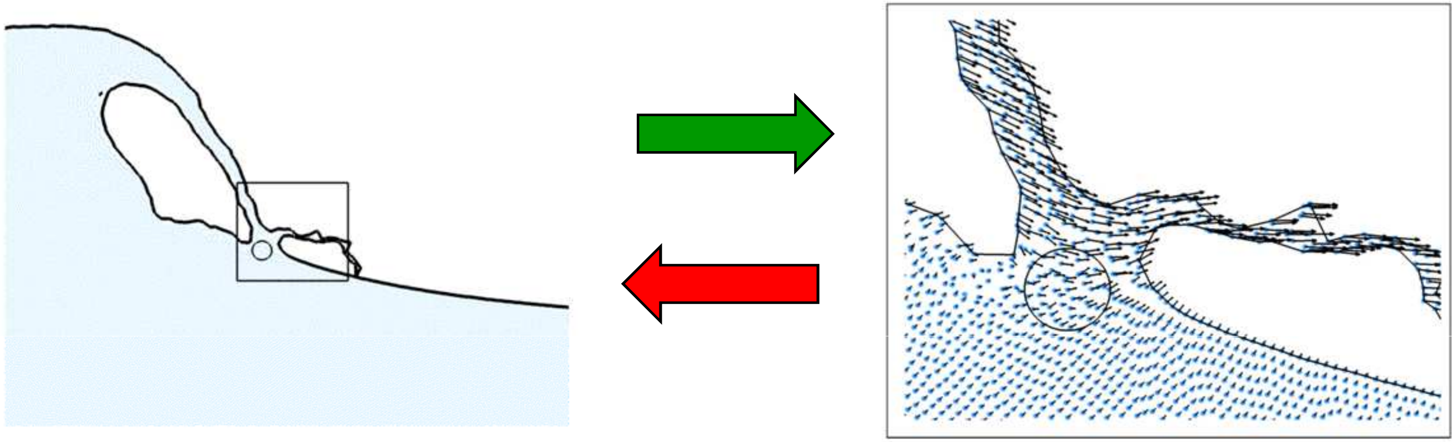
5. A. Colagrossi, M. Antuono, A. Souto-Iglesias, D. Le Touzé, *Theoretical Analysis and numerical verification of the consistency of viscous SPH formulation in simulating free-surface flows,* Physical Review E, 84, 026705, August, 2011.

6. A. Colagrossi, A. Souto-Iglesias, M. Antuono, S. Marrone, *Smoothed-particle-hydrodynamics modeling of dissipation mechanisms in gravity waves*, Physical Review E, 87, 023302, 2013

7. D. Le Touzé, A. Colagrossi, G. Colicchio, M. Greco, *A critical investigation of Smoothed Particle Hydrodynamics applied to problems with free surfaces*, 73, 660-691, International Journal of Numerical Methods in Fluids, 2013.

Smoothed Particle Hydrodynamics Method

Two approaches of derivation



1) From Continuum to Discrete Level (Discretization of PDE)

2) From the Discrete Level to Continuum

K. Oelschliiger, **On the connection between Hamiltonian many-particle systems and the hydrodynamical equations**, *Arch. Rat. Mech. An.* 115, 297 (1991).

E. Tonti**, Why starting from differential equations for computational physics?,** *Journal of Computational Physics*, 257, 1260–1290, (2014).

Smoothed Particle Hydrodynamics Method

Two approaches of derivation

E. Tonti / Journal of Computational Physics 257 (2014) 1260–1290

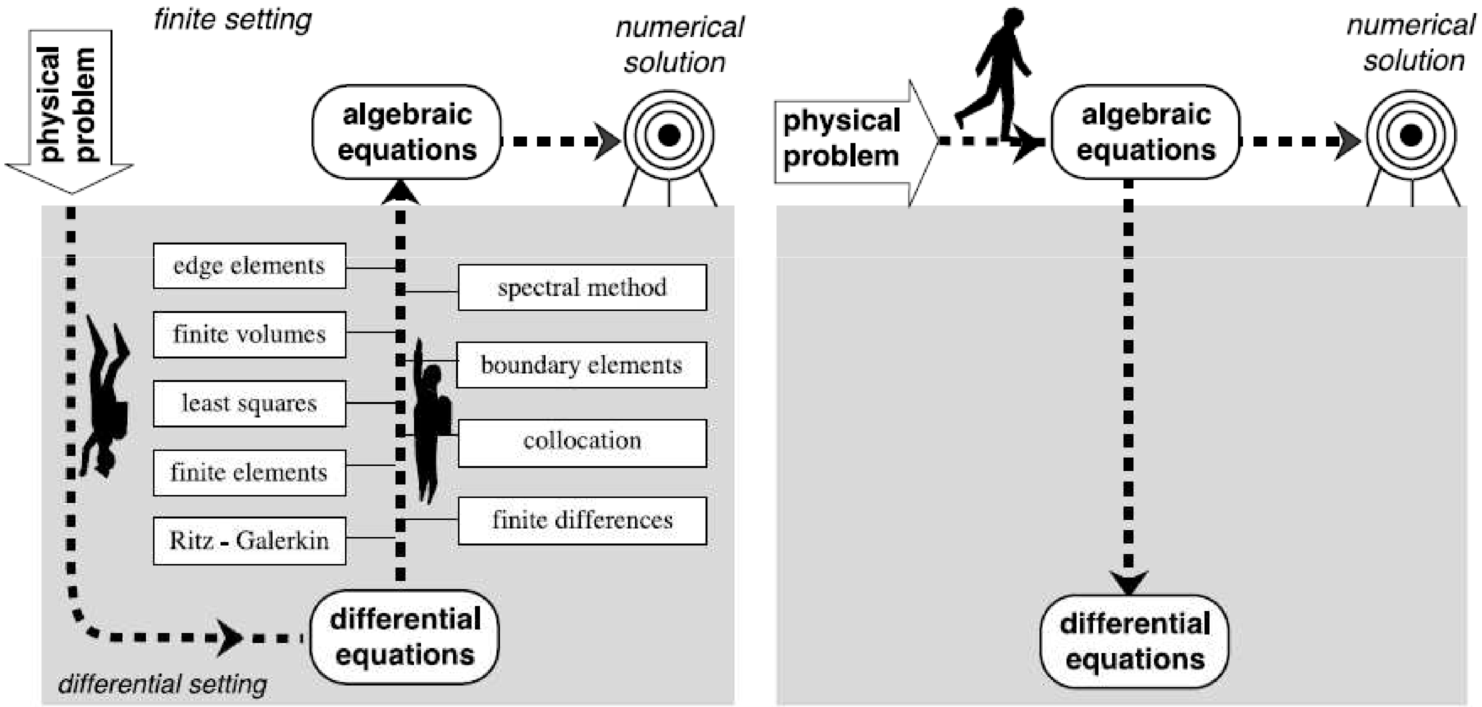
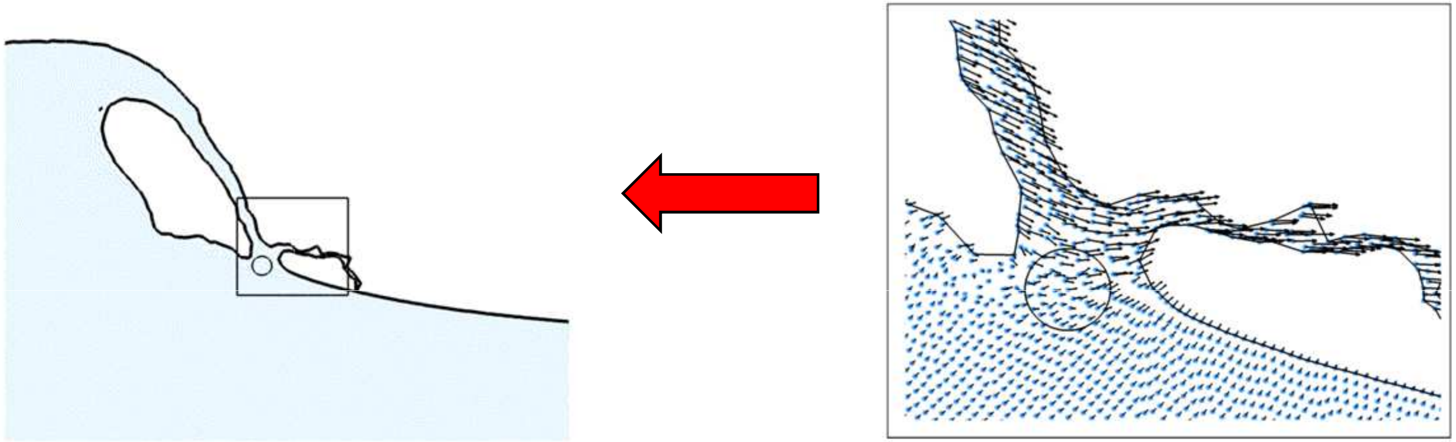


Fig. 1. (left) The tortuous path to obtain a numerical solution to a physical problem; (right) the direct procedure.

Smoothed Particle Hydrodynamics Method

Two approaches of derivation



1) From Continuum to Discrete Level, (Discretization of PDE)

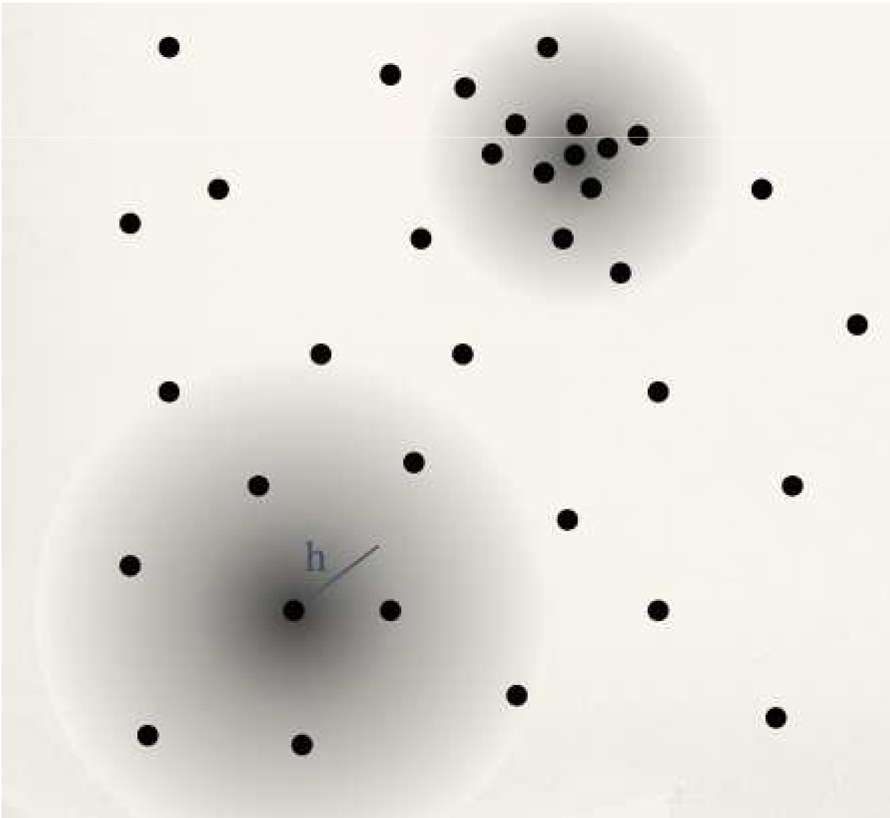
2) From the Discrete Level to Continuum

K. Oelschliiger, **On the connection between Hamiltonian many-particle systems and the hydrodynamical equations**, *Arch. Rat. Mech. An.* 115, 297 (1991).

SPH: From the Discrete Level to Continuum

Example: Hamiltonian System of interacting particles

8th International SPHERIC Workshop in Trondheim, June 2013.
"Particles for fluids: SPH methods as a mean-field flow",
Dr Daniel Price, Monash University, Australia



Density Estimation

$$\rho(r) = \sum_j m_j W(r_j - r; h)$$

Lucy, L. (1977). **A Numerical Approach to the Testing of Fission Hypothesis**. *The Astronomical Journal* 82 (12), 1013–1024.

SPH: From the Discrete Level to Continuum

Example: Hamiltonian System of interacting particles

Density Estimation (with constant h)

$$\rho(r) = \sum_j m_j W(r - r_j)$$

$$M = \sum_j m_j$$

Particle masses do not change in time

$$\nabla\rho(r) = \sum_j m_j \frac{\partial W(r_j - r)}{\partial r}$$

$$\frac{D\rho_i}{Dt} = -\sum_j m_j \left(u_j - u_i\right) \bullet \nabla_i W_{ij}$$

Galilean invariance is respected

SPH: From the Discrete Level to Continuum

Example: Hamiltonian System of interacting particles

$$L = \sum_i m_i \left(\frac{u_i^2}{2} - e_i \right)$$

$$\delta e_i = \cancel{T_i \delta S_i} - p_i \delta v_i \quad \Longrightarrow \quad \delta e_i = + \frac{p(\rho_i)}{\rho_i} \left(\frac{\delta \rho_i}{\rho_i} \right)$$

$$e(\rho_i) - e(\rho_0) = \int_{\rho_0}^{\rho_i} \frac{p(\rho)}{\rho^2} d\rho \quad \Longleftarrow \quad \left. \frac{de}{d\rho} \right|_i = \frac{p(\rho_i)}{\rho_i^2}$$

SPH: From the Discrete Level to Continuum

Example: Hamiltonian System of interacting particles

$$L = \sum_i m_i \left[u_i^{\,2}/2 - e(\rho_i) \right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u_i}\right) - \frac{\partial L}{\partial r_i} = 0$$

$$\sum_i m_i \frac{du_i}{dt} - \sum_i m_i \left(-\frac{p_i}{\rho_i^2} \right) \sum_j m_j \nabla_i W_{ij} = 0$$

$$\nabla_i W_{ij} = -\nabla_j W_{ij}$$

$$m_i \frac{du_i}{dt} = -m_i \sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla_i W_{ij}$$

The Linear and Angular momenta of the particle system is exactly preserve !

SPH: From the Discrete Level to Continuum

Example: Hamiltonian System of interacting particles

$$E_K = \frac{1}{2}\sum_i m_i u_i^{\,2} \qquad E_k + E_i = \text{const} \qquad \dot{E}_k + \dot{E}_i = 0$$

$$E_i = \sum_i m_i e_i$$

$$\frac{D\rho_i}{Dt} = -\sum_j m_j \left(u_j - u_i\right)\cdot \nabla_i W_{ij} \qquad \sum_i m_i u_i \cdot \frac{du_i}{dt} + \sum_i m_i \left(-\frac{p_i}{\rho_i^2}\right)\left(\frac{D\rho_i}{Dt}\right) = 0$$

$$\nabla_i W_{ij} = -\nabla_j W_{ij}$$

$$\sum_i m_i u_i \bullet \frac{du_i}{dt} = \sum_i m_i u_i \bullet \sum_j \left[-m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2}\right)\nabla_i W_{ij} \right]$$

SPH: From the Discrete Level to Continuum

Example: Hamiltonian System of interacting particles

$$\rho_i \frac{du_i}{dt} = -\rho_i \sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla_i W_{ij}$$

$$\rho_i = \sum_j m_j W_{ij}$$

$$\Downarrow$$

$$\frac{D\rho_i}{Dt} = -\sum_j m_j \left(u_j - u_i \right) \bullet \nabla_i W_{ij}$$

$$N \to \infty$$

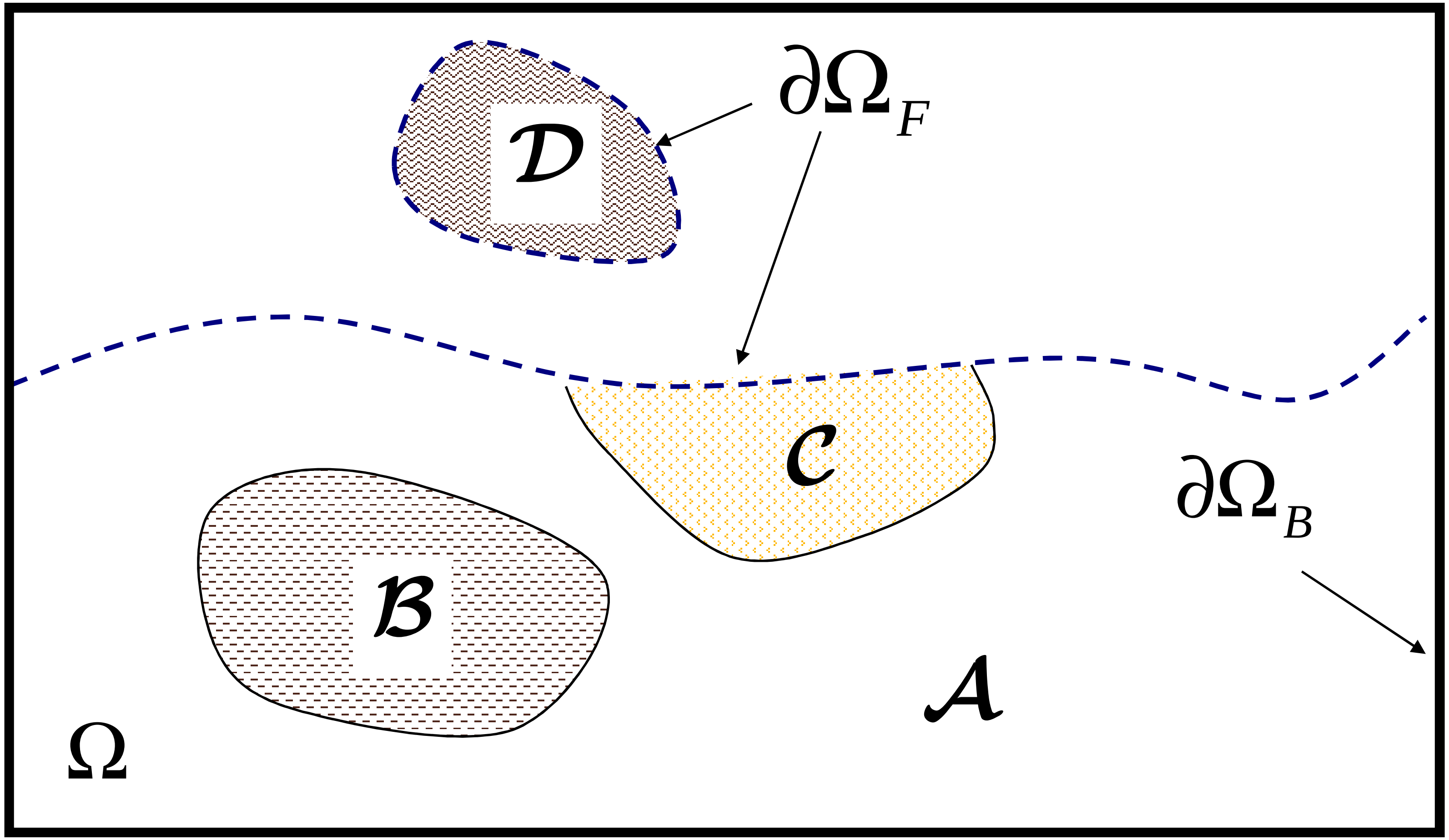
$$W_{ij} \to \delta_{ij}$$

$$\begin{cases} \dfrac{D\rho}{Dt} = -\rho \mathrm{Div}(u) \\ \rho \dfrac{Du}{Dt} = -\nabla p \\ p = f(\rho) \\ \dfrac{De}{Dt} = \left(-\dfrac{p}{\rho^2} \right) \dfrac{D\rho}{Dt} \end{cases}$$

Boundary Conditions ????

Boundary Conditions

Kernel Truncation



Free-surface Boundary Conditions
Kinematic and Dynamic conditions

- **Kinematic Boundary condition:**

Material points on the free surface remain on it during their evolution (absence of discontinuous events like fluid-fluid/ fluid-solid impacts)

- **Dynamic Boundary condition:**

Free surface is a free-stress surface

$$\mathbb{T}\boldsymbol{n} = [-p + \lambda \operatorname{div}(\boldsymbol{u})]\,\boldsymbol{n} + \mu\,(\boldsymbol{n} \times \omega) + 2\,\mu\,\nabla\boldsymbol{u}\,\boldsymbol{n} \quad \mathbf{=0}$$

$$\begin{cases} p = \lambda \operatorname{div}(\boldsymbol{u}) + 2\mu \dfrac{\partial \boldsymbol{u}}{\partial n} \cdot \boldsymbol{n}_F \\ \\ \omega \cdot (\boldsymbol{\tau}_F \times \boldsymbol{n}_F) = -2 \dfrac{\partial \boldsymbol{u}}{\partial n} \cdot \boldsymbol{\tau}_F \end{cases} \qquad \forall \boldsymbol{r} \in \partial\Omega_F$$

SPH: From the Discrete Level to Continuum

Example: PDE for inviscid isentropic flow

$$\begin{cases} \dfrac{D\rho}{Dt} = -\rho \mathrm{Div}(u) \\ \rho \dfrac{Du}{Dt} = -\nabla p \\ p = f(\rho) \\ \dfrac{De}{Dt} = \left(-\dfrac{p}{\rho^2} \right) \dfrac{D\rho}{Dt} \end{cases}$$

Continuity equation

Momentum conservation

Equation of state

Energy equation

Lagrangian for compressible, non dissipative flow (Eckart 1960):

$$L = \int_{\Omega} \rho \left(\frac{u^2}{2} - e(\rho) \right) dV$$

SPH scheme N°1

Hamiltonian System of interacting particles

$$\begin{cases} \rho_i = \sum_j m_j W_{ij}; \quad p_i = f(\rho_i) \\ \dfrac{D\mathrm{u}_i}{Dt} = -\sum_{j=1}^{N} \left(\dfrac{p_i}{\rho_i^2} + \dfrac{p_j}{\rho_j^2}\right) \nabla_i W_{ij} \; m_j \\ \dfrac{D\mathrm{r}_i}{Dt} = \mathrm{u}_i \end{cases}$$

$$\begin{cases} \rho_i = \Phi(r) \\ \dfrac{D\mathrm{u}_i}{Dt} = -\psi(r) \\ \dfrac{D\mathrm{r}_i}{Dt} = \mathrm{u}_i \end{cases}$$

Initial conditions consistent with the density estimator:

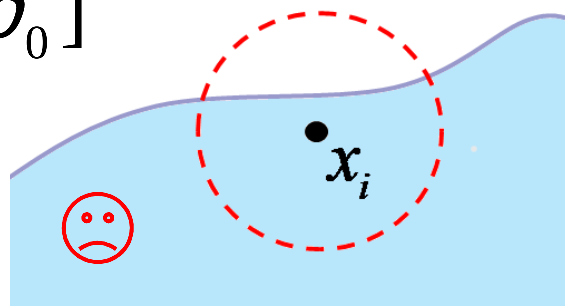
$$(r_{i0}, \; u_{i0})$$

$$\rho_{i0} = f^{-1}(p_{i0})$$

$$\sum_j m_j W_{ij} = \rho_{i0}$$

$$[m] = [W]^{-1} [\rho_0]$$

"Volume Matrix"



SPH scheme N°1

Particle Volumes

In Scheme N°1 it is not necessary to introduce the concept of particle volumes. Anyway one can define:

$$V_i = \frac{m_i}{\rho_i} = \frac{m_i}{\sum_j m_j W_{ij}}$$

$$V_{i0} = \frac{m_i}{\rho_{i0}} = \frac{m_i}{\sum_j m_j W_{ij}}$$

$$V_i(t) = \frac{m_i}{\rho_i(t)}$$

$$V_0 \neq \sum_i V_{i0}$$

SPH does not guarantee that the particles occupy the right geometrical volume.

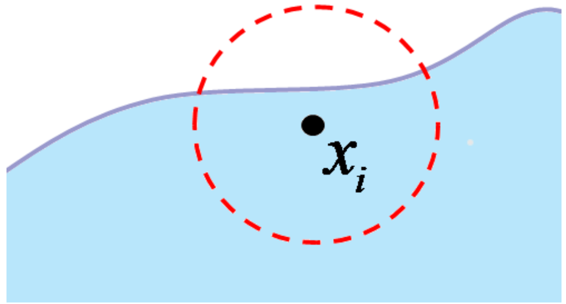
SPH schemes

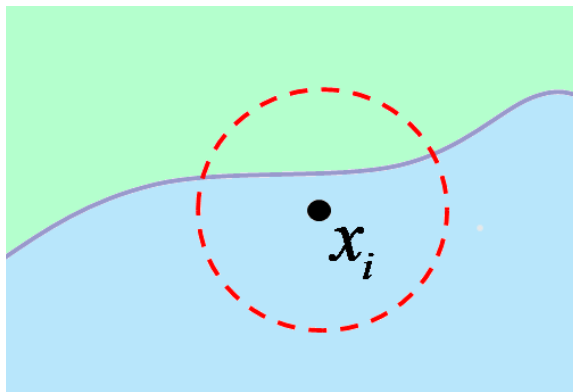
SPH formulation through Volume Estimation:

$$\rho_i = \sum_j m_j W_{ij}$$

☹ **Free-surface flows**

☹ **Multi-fluids flows**





SPH scheme N°2

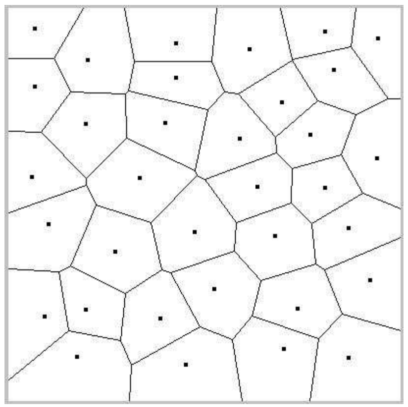
"Hamiltonian System" of interacting particles

$$\begin{cases} \dfrac{D\rho_i}{Dt} = \sum_{j=1}^{N} (\mathrm{u}_i - \mathrm{u}_j) \cdot \nabla W_{ij}\ m_j;\ p_i = f(\rho_i) \\ \dfrac{D\mathrm{u}_i}{Dt} = -\sum_{j=1}^{N} \left(\dfrac{p_i}{\rho_i^2} + \dfrac{p_j}{\rho_j^2}\right) \nabla_i W_{ij}\ m_j \\ \dfrac{D\mathrm{r}_i}{Dt} = \mathrm{u}_i \end{cases}$$

$$\begin{cases} \dfrac{D\rho_i}{Dt} = F(r,u) \\ \dfrac{D\mathrm{u}_i}{Dt} = -G(r,\rho) \\ \dfrac{D\mathrm{r}_i}{Dt} = \mathrm{u}_i \end{cases}$$

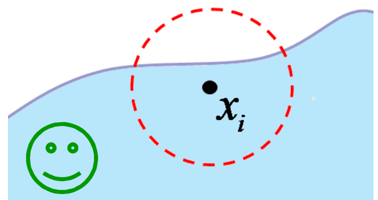
$$\rho_{i0} = f^{-1}(p_{i0})$$

$$(r_{i0},\ u_{i0}, \rho_{i0})$$



$$m_i = \rho_{i0} V_{i0}$$

$$V_i(t) = \frac{m_i}{\rho_i(t)}$$



SPH schemes

SPH formulation through Volume Estimation:

$$\rho_i = \sum_j m_j W_{ij}$$

☹ **Free-surface flows**

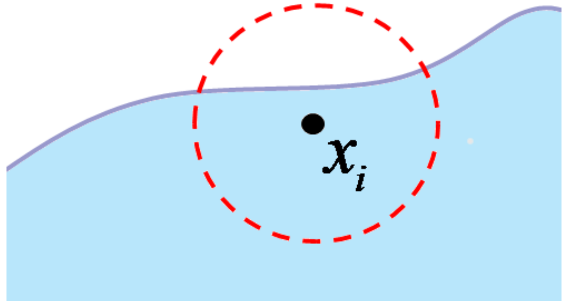
☹ **Multi-fluids flows**

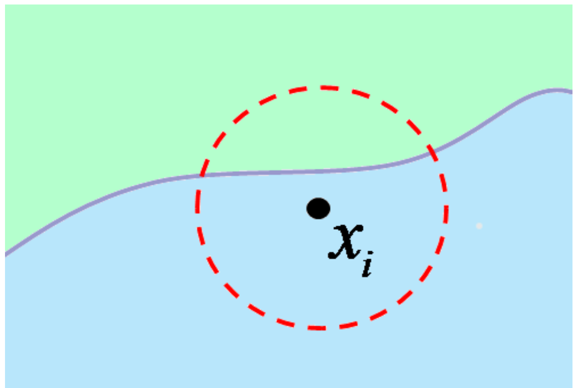
$$\frac{D\rho_i}{Dt} = -\sum_j m_j \left(u_j - u_i\right) \bullet \nabla_i W_{ij}$$

$$< \nabla p > = -\rho_i \sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla_i W_{ij}$$

☺ **Free-surface flows**

☹ **Multi-fluids flows**





SPH: From the Discrete Level to Continuum

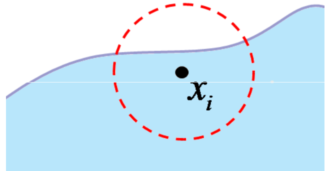
Isentropic flow: Equation of state

$$p(\rho) = K\rho^{\gamma}$$ **Politropic law** $$p(\rho) = \frac{c_0^2 \rho_0}{\gamma}\left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

Liquid: weakly compressible regime $$\left(\frac{\rho}{\rho_0} - 1\right) = \varepsilon \leq 10^{-2}$$

$$P_0 = 0$$

$$p(\rho) = \frac{c_0^2 \rho_0}{\gamma}(1+\varepsilon)^{\gamma} = \frac{c_0^2 \rho_0}{\gamma} + c_0^2 \rho_0 \varepsilon + o(\varepsilon) \approx P_0 + c_0^2 \rho_0 \varepsilon$$



The adiabatic index γ, which is an important parameter for gaseous phases, has a negligible role for liquid phases.

$$e(\rho) - e(\rho_0) = \int_{\rho_0}^{\rho_i} \frac{p(\rho)}{\rho^2} d\rho \qquad e(\rho) - e(\rho_0) = \begin{cases} \frac{c_0^2}{\gamma(\gamma-1)}\left[\left(\frac{\rho}{\rho_0}\right)^{\gamma} - 1\right] & \gamma \neq 1 \\ c_0^2 \ln\left(\frac{\rho}{\rho_0}\right) & \gamma = 1 \end{cases}$$

$$\gamma \neq 1 \qquad \mathrm{p} = (\gamma - 1)e\rho$$ **Ideal gas law**

SPH: From the Discrete Level to Continuum

Equation of state for free-surface flow

$$p(\rho) = K\rho^{\gamma} - P_0$$

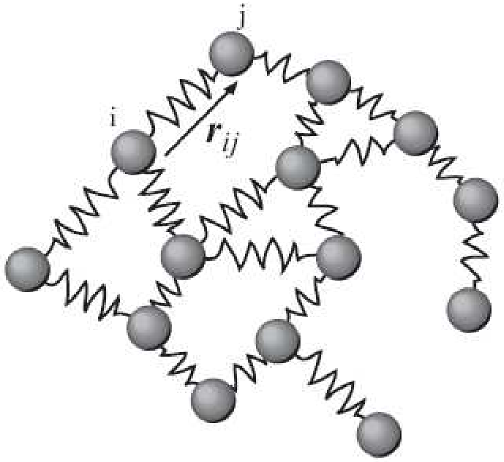
$$p(\rho) = \frac{c_0^2 \rho_0}{\gamma}\left[\left(\frac{\rho}{\rho_0}\right)^{\gamma} - 1\right]$$

For water this EoS is called Stiffned EoS (γ=6.1 - 7 and c_0 = 1497 m/s)

weakly compressible regime $\left(\frac{\rho}{\rho_0} - 1\right) = \varepsilon \le 10^{-2}$

$$p(\rho) = \frac{c_0^2 \rho_0}{\gamma}\varepsilon^{\gamma} \approx c_0^2 \rho_0 \varepsilon$$

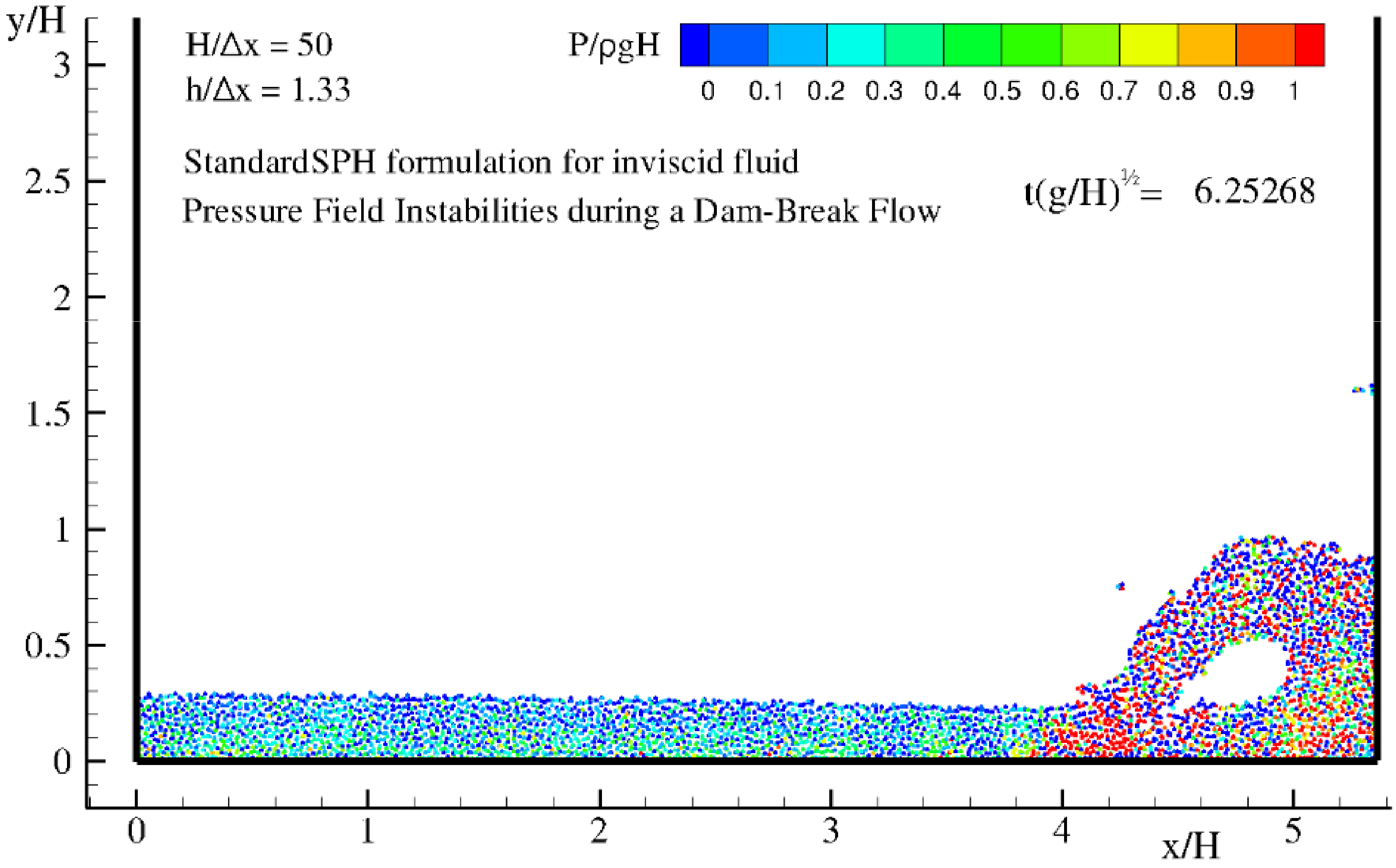
$$e(\rho) - e(\rho_0) = c_0^2 \left[\ln\left(\frac{\rho}{\rho_0}\right) - \frac{\rho_0}{\rho}\right] \approx c_0^2 \left(1 + \frac{\varepsilon^2}{2}\right)$$



$$e(\rho) = \frac{1}{2} c_0^2 \varepsilon^2$$

Smoothed Particle Hydrodynamics Method

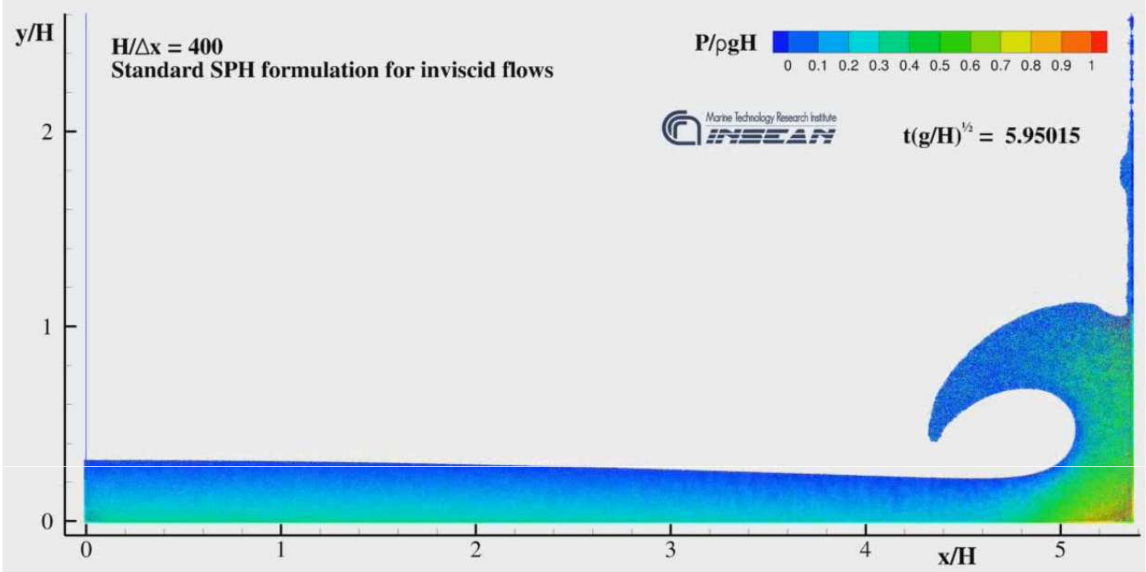
Inviscid Free-surface flows

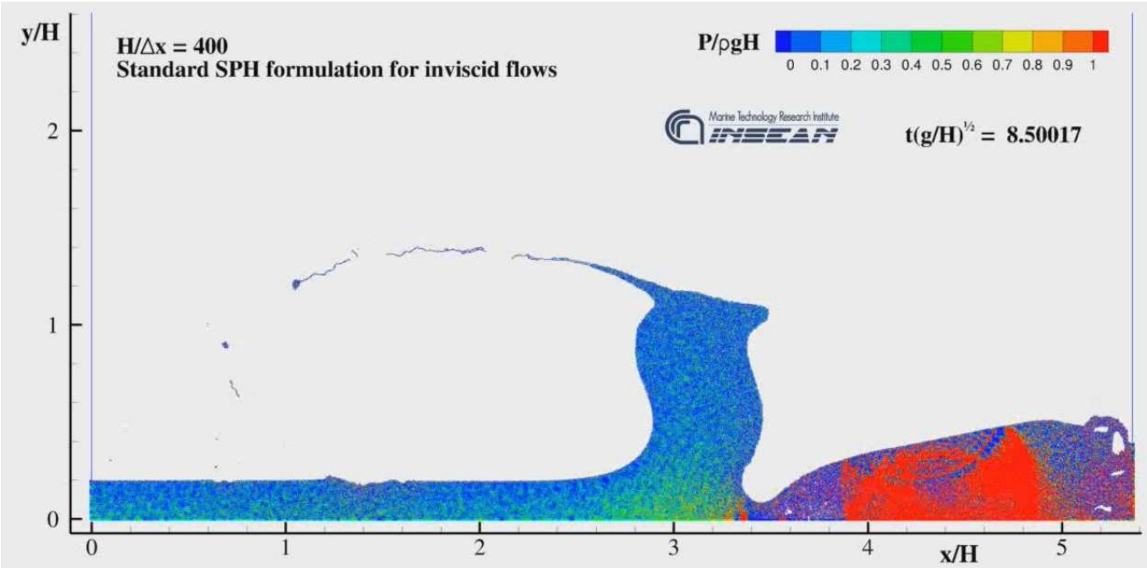


Smoothed Particle Hydrodynamics Method

Inviscid Free-surface flows

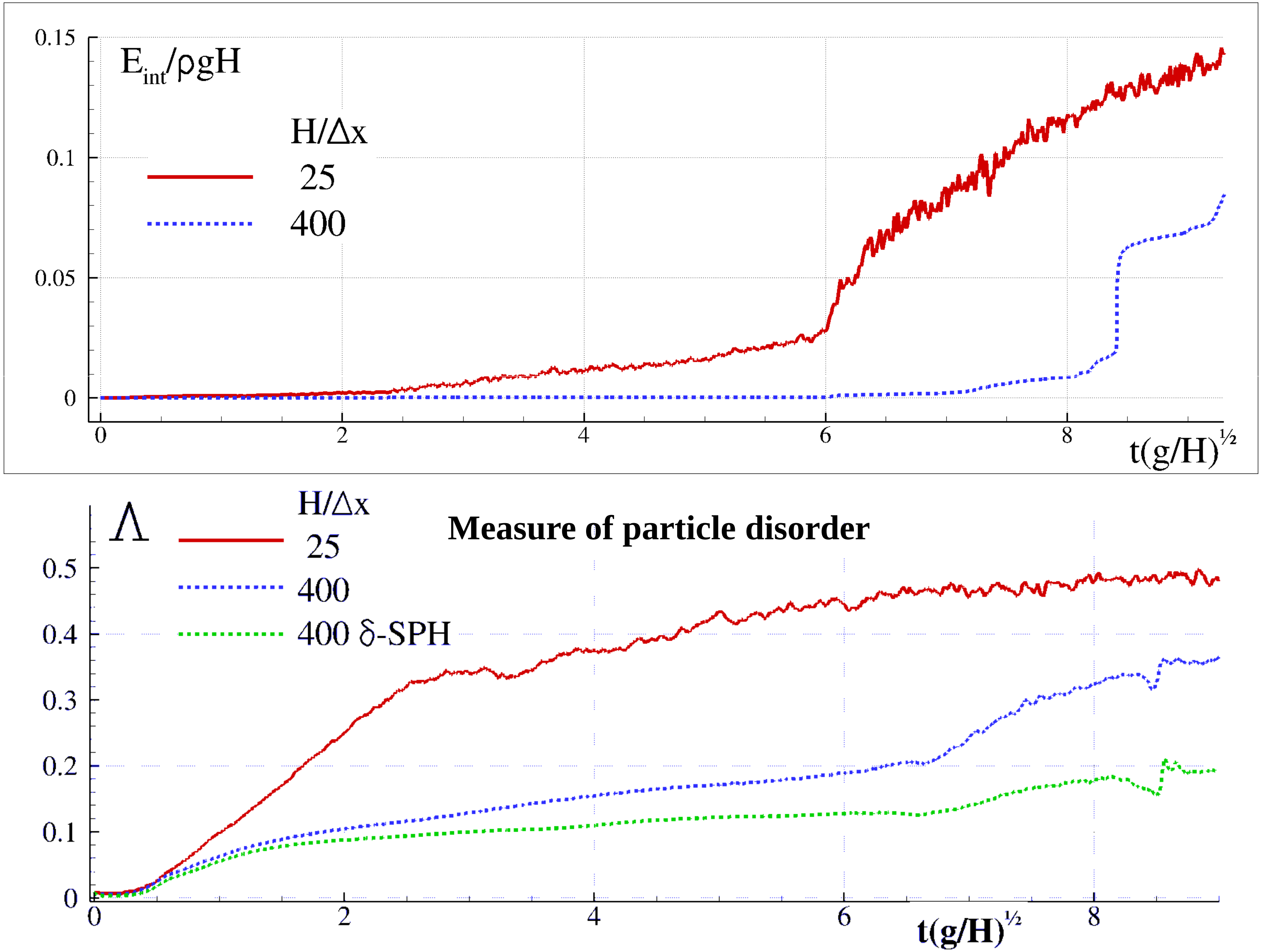
Standard SPH
High Spatial Resolution





Smoothed Particle Hydrodynamics Method

Inviscid Free-surface flows



Smoothed Particle Hydrodynamics Method

Inviscid Free-surface flows

M. Antuono et al. , *Energy conservation in the -SPH scheme*, 9th SPHERIC Workshop, Paris, (2014)
13:15 Wednesday, Session 8 – Turbulence, Structures, Energy

S. Marrone et al., *On the model inconsistencies in simulating breaking wave with mesh-based and particle methods*, 9th SPHERIC Workshop, Paris, (2014)
09:00 Thursady, Session 11 – Water Waves

A.Souto-Iglesias et al., *Energy decomposition analysis in free-surface flows: road-map for the direct computation of wave breaking dissipation*, 9th SPHERIC Workshop, Paris, (2014)
13:55 Thursady, Session 14 – Free-Surface Flow

SPH scheme N°3

Volume estimator – derivation

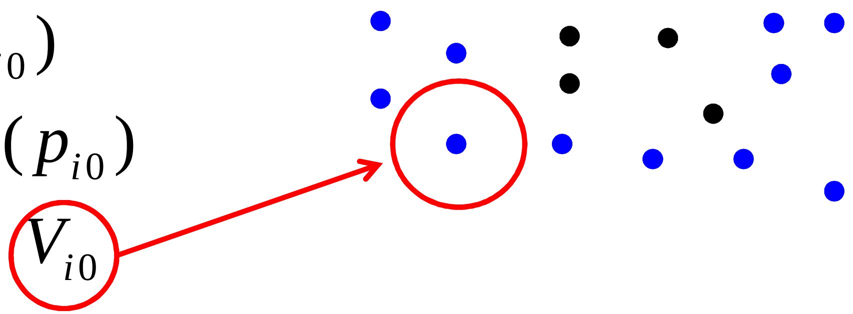
$$\begin{cases} V_i = \dfrac{1}{\sum_j W_{ij}};\ \rho_i = \dfrac{m_i}{V_i};\ p_i = f(\rho_i) \\ m_i \dfrac{D\mathbf{u}_i}{Dt} = -\sum_{j=1}^{N} (p_i V_i^2 + p_j V_j^2) \nabla_i W_{ij} \\ \dfrac{D\mathbf{r}_i}{Dt} = \mathbf{u}_i \end{cases}$$

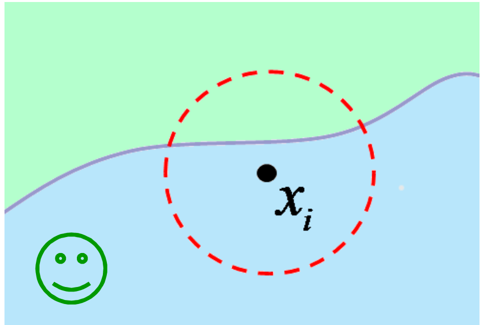
$$\begin{cases} V_i = \Phi(r) \\ \dfrac{D\mathbf{u}_i}{Dt} = -\Psi(r) \\ \dfrac{D\mathbf{r}_i}{Dt} = \mathbf{u}_i \end{cases}$$

$$(r_{i0},\ u_{i0})$$

$$\rho_{i0} = f^{-1}(p_{i0})$$

$$m_i = \rho_{i0} V_{i0}$$





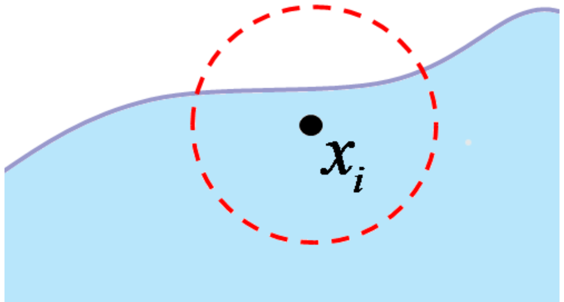
SPH schemes

SPH formulation through Volume Estimation:

$$\rho_i = \sum_j m_j W_{ij}$$

☹ **Free-surface flows**

☹ **Multi-fluids flows**



$$\frac{D\rho_i}{Dt} = -\sum_j m_j \left(u_j - u_i\right) \bullet \nabla_i W_{ij}$$

$$< \nabla p > = -\rho_i \sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla_i W_{ij}$$

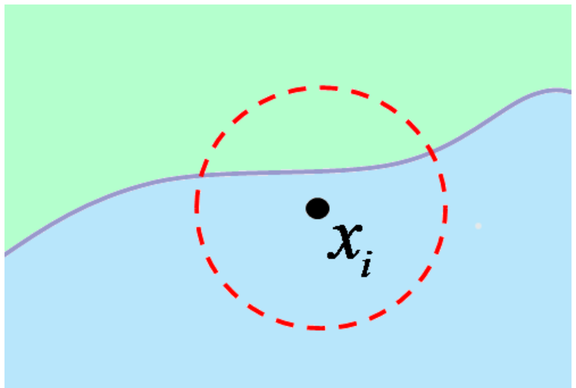
☺ **Free-surface flows**

☹ **Multi-fluids flows**

$$V_i = \frac{1}{\sum_j W_{ij}}$$

☹ **Free-surface flows**

☺ **Multi-fluids flows**



$$V_0 \neq \sum_i V_{i0}$$

SPH scheme N°4

Time-Volume estimation – derivation

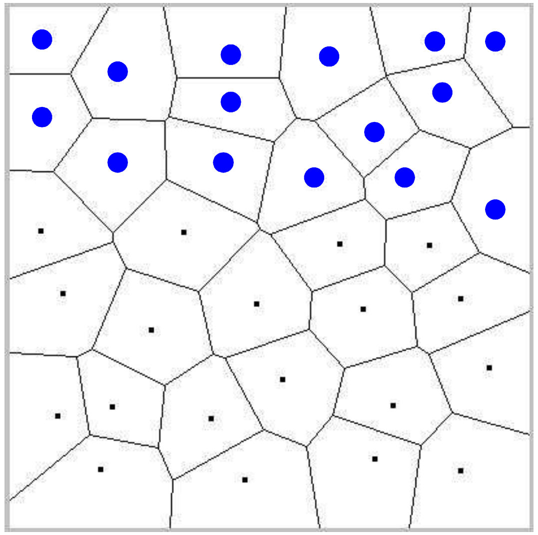
$$\begin{cases} \dfrac{DV_i}{Dt} = V_i^2 \sum_{j=1}^{N} (\mathrm{u}_i - \mathrm{u}_j)\cdot \nabla W_{ij} \;;\; \rho_i = \dfrac{m_i}{V_i};\; p_i = f(\rho_i) \\ m_i \dfrac{D\mathrm{u}_i}{Dt} = -\sum_{j=1}^{N} (p_i V_i^2 + p_j V_j^2) \nabla_i W_{ij} \\ \dfrac{D\mathrm{r}_i}{Dt} = \mathrm{u}_i \end{cases}$$

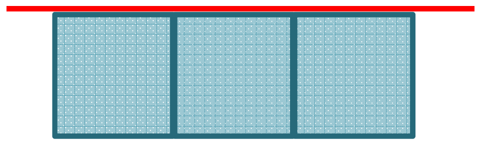
$$\begin{cases} \dfrac{DV_i}{Dt} = F(r,u) \\ \dfrac{D\mathrm{u}_i}{Dt} = -G(r,V) \\ \dfrac{D\mathrm{r}_i}{Dt} = \mathrm{u}_i \end{cases}$$

$$(r_{i0},\; u_{i0},\; V_{i0})$$

$$\rho_{i0} = f^{-1}(p_{i0})$$

$$m_i = \rho_{i0}\; V_{i0}$$





☺ **Free-surface flows**

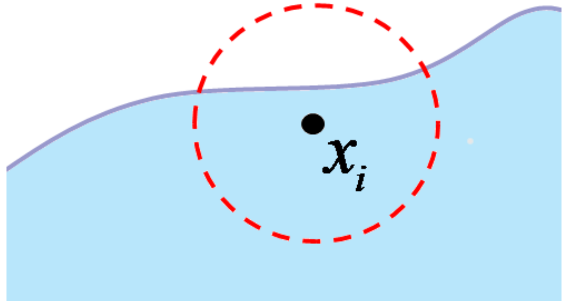
☺ **Multi-fluids flows**

SPH schemes

SPH formulation through Volume Estimation:

$$\rho_i = \sum_j m_j W_{ij}$$

- Free-surface flows (sad face)
- Multi-fluids flows (sad face)



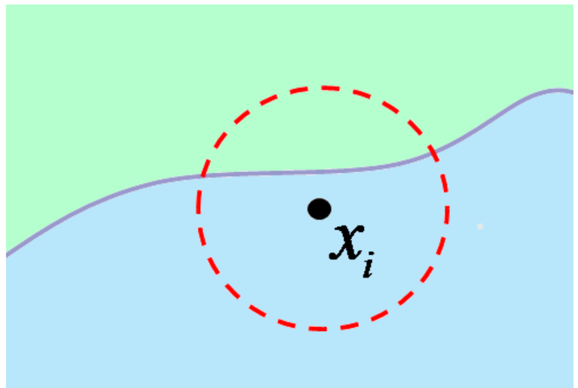
$$\frac{D\rho_i}{Dt} = -\sum_j m_j \left(u_j - u_i\right) \bullet \nabla_i W_{ij}$$

$$\nabla p = -\rho_i \sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla_i W_{ij}$$

- Free-surface flows (happy face)
- Multi-fluids flows (sad face)

$$V_i = \frac{1}{\sum_j W_{ij}}$$

- Free-surface flows (sad face)
- Multi-fluids flows (happy face)



$$\frac{DV_i}{Dt} = -V_i^2 \sum_j \left(u_j - u_i\right) \cdot \nabla_i W_{ij}$$

- Free-surface flows (happy face)
- Multi-fluids flows (happy face)

SPH scheme N°4 bis

Evaluation of density through Shepard Formula

$$\frac{DV_i}{Dt} = -V_i^2 \sum_j \left(u_j - u_i\right) \cdot \nabla_i W_{ij}$$

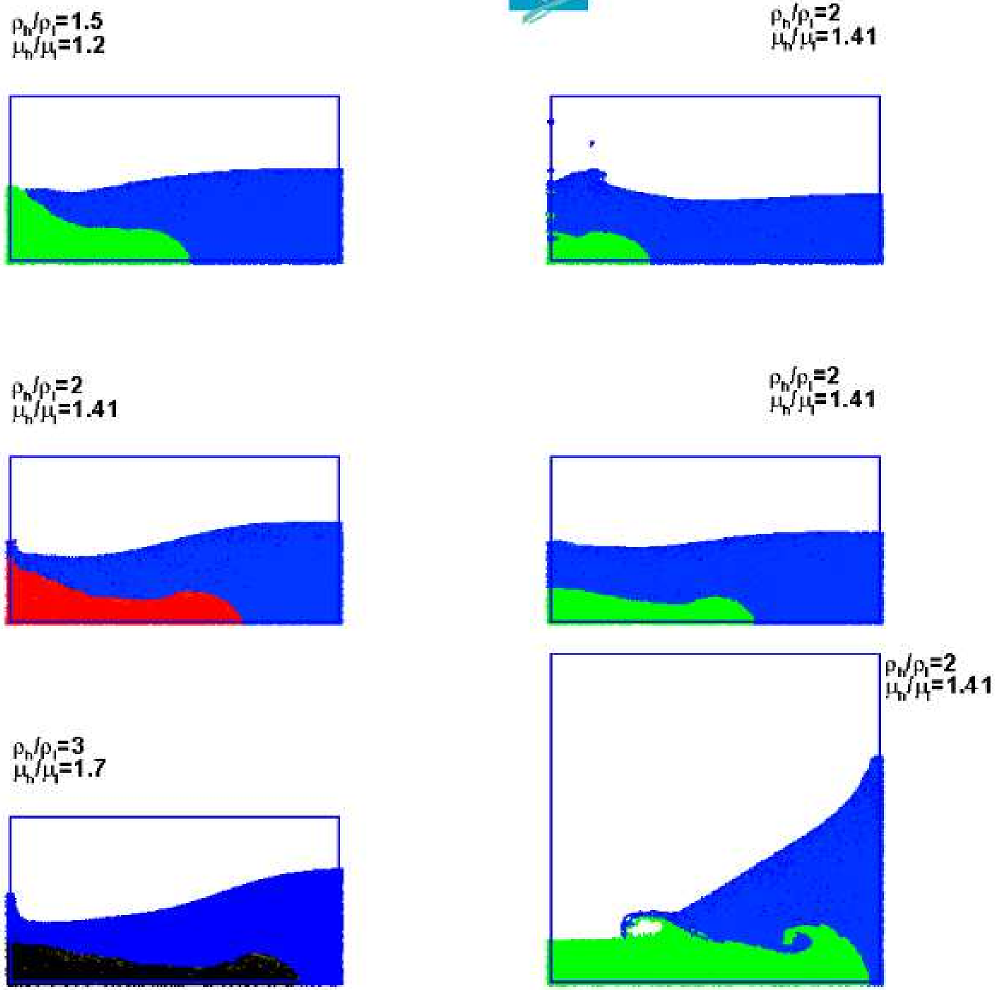
$$W_j^S(x) = \frac{W_j(x)}{\sum_{k \in \chi} W_k(x) V_k}$$ Shepard Kernel

$$\sum_j W_{ij}^S V_j = 1 \qquad \rho_i = \sum_{j \in \chi} m_j W_{ij}^S$$

Grenier, N, Antuono, M, Colagrossi, A, Le Touzé, D, Alessandrini, B, *An Hamiltonian interface SPH formulation for multi-fluid and free surface flows*, JOURNAL OF COMPUTATIONAL PHYSICS Volume: 228 Issue: 22, Pages: 8380-8393 Published: 2009

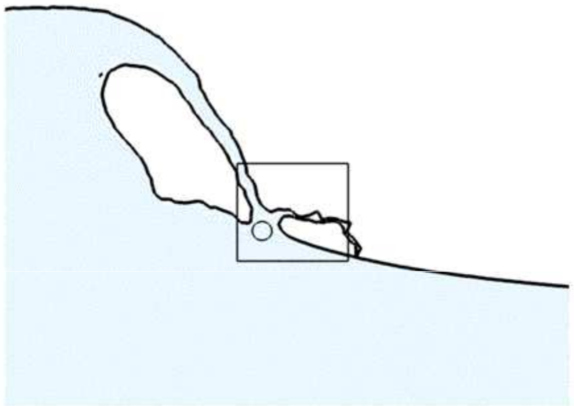
SPH scheme N°4 bis

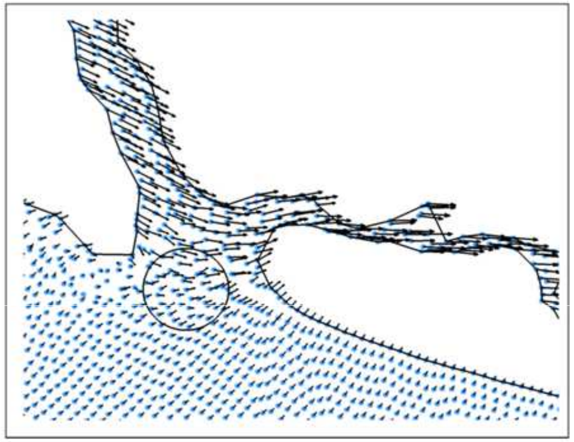
Evaluation of density through Shepard Formula



Smoothed Particle Hydrodynamics Method

Two approaches of derivation





1) From Continuum to Discrete Level, (Discretization of PDE)

2) From the Discrete Level to Continuum

Mollified Navier-Stokes equations

Integral Interpolation

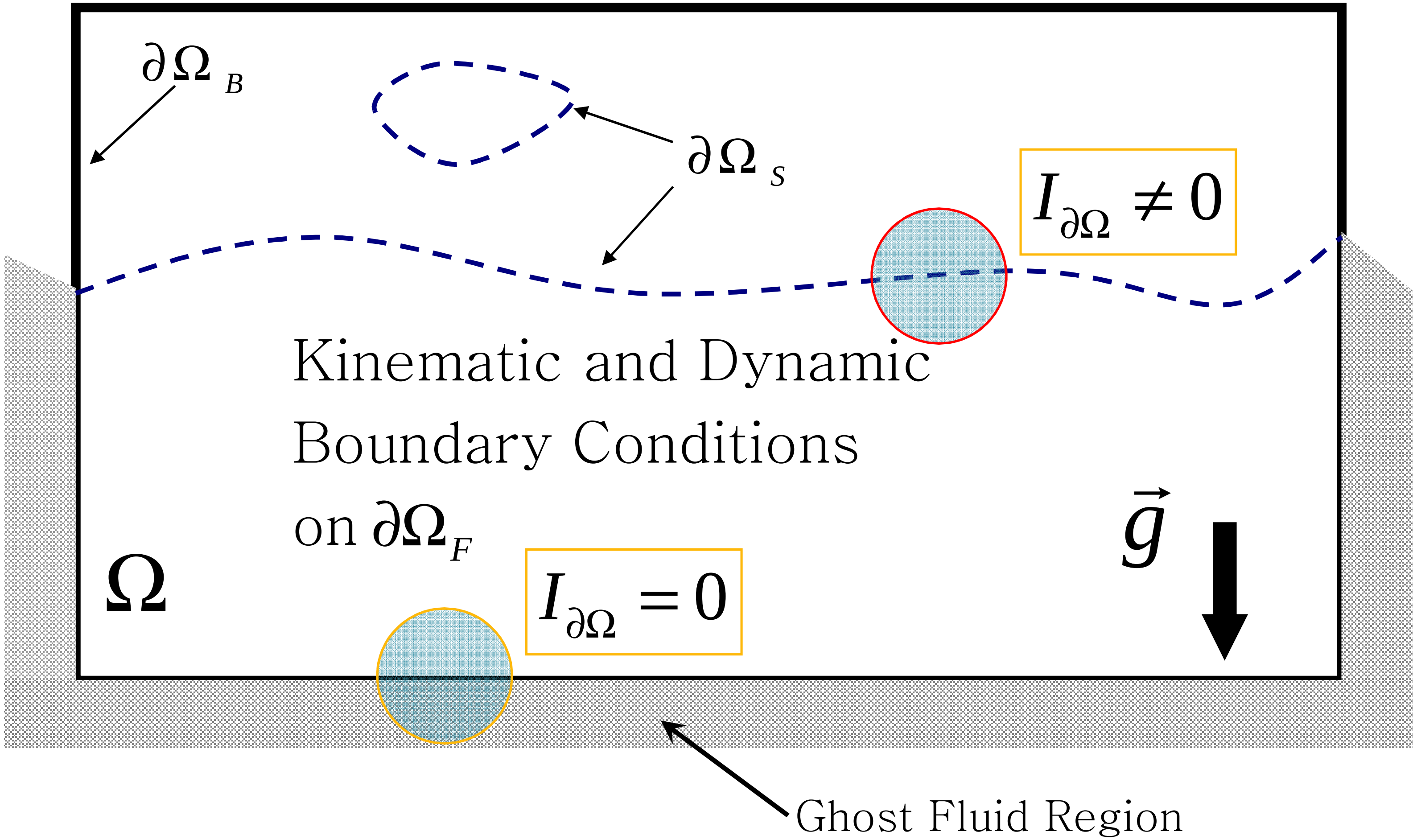
$$\begin{cases} \dfrac{D\rho}{Dt} = -\rho < \nabla \cdot \mathbf{u} >; \quad p = c_0^2(\rho - \rho_0) \\ \rho \dfrac{D\mathbf{u}}{Dt} = -< \nabla p > + < \nabla \cdot V > + \rho \mathbf{g} \end{cases}$$

$$< \nabla f(x) > = \int_\Omega f(x^*) \nabla W(x - x^*; h)\, dV^* + I_{\partial\Omega}$$

$$< \nabla^2 f(x) > = \int_\Omega f(x^*) \nabla^2 W(x - x^*; h)\, dV^* + I'_{\partial\Omega}$$

Kernel truncation

Solid boundaries & Ghost fluid approach



Smoothed differential operators

Galilean Invariance

$$<\nabla u(r)> = \int_\Omega u(r^*) \otimes \nabla W(r - r^*; h)\, dV^* + I_{\partial\Omega}$$

$<\nabla u(r)> = O(1/h)$

close to the free-surface

$$w \longrightarrow u + [c + \Omega(r - r_0)]$$

$$I_2 = \int_\Omega \nabla W(r - r^*; h)\, dV^* = 0$$

$$I_4 = \mathbf{I} - \int_\Omega r^* \otimes \nabla W(r - r^*; h)\, dV^* = 0$$

$$<\nabla w(r)> = <\nabla u(r)> + \Omega$$

Close to the free surface I_2 and I_4 are not zero and the Galilean Invariance is not respected anymore.

Smoothed differential operators

Galilean Invariance

$$< \nabla u(r) > = \int_{\Omega} \left[u(r^*) - u(r) \right] \otimes \nabla W(r - r^*; h)\, dV^*$$

The so-called antisymmetric form is ok for translation but still presents errors when considering rotation (I4 is not zero), however :

$$< Div(u)(r) > = \int_{\Omega} \left[u(r^*) - u(r) \right] \cdot \nabla W(r - r^*; h)\, dV^*$$

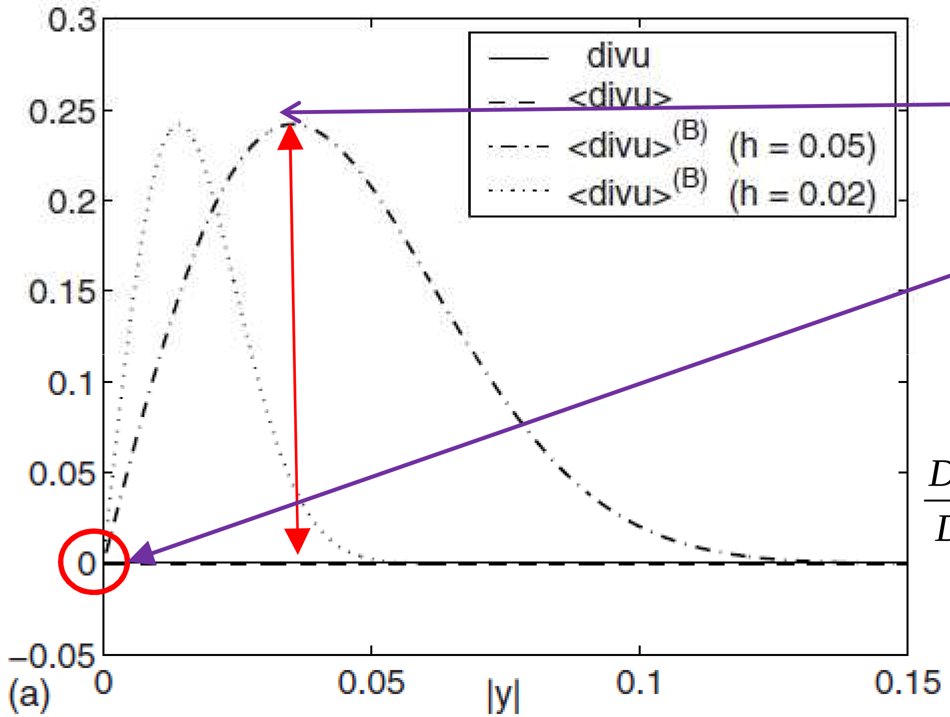
$< \nabla \cdot u(r) > = < \nabla \cdot w(r) >$ it is always guaranteed

$< \nabla \cdot u(r) > = O(h)$ Close to the free-surface (in a weak sense)

Colagrossi, M. Antuono, D. Le Touzé, **Theoretical considerations on the free-surface role in the smoothed-particle-hydrodynamics model**, *Physical Review E*, 79, 1-13, 2009.

Smoothed differential operators

Consistency of the velocity divergence operator



Local inconsistency

$Div(u) = 0$ at the free surface

$$\frac{D\rho}{Dt} = 0 \quad \Rightarrow \rho = \rho_0 \Rightarrow p = 0 \text{ at the free surface}$$

Only true for inviscid flows

Colagrossi, M. Antuono, D. Le Touzé, **Theoretical considerations on the free-surface role in the smoothed-particle-hydrodynamics model**, *Physical Review E*, 79, 1-13, 2009.

Smoothed differential operators

Energy conservation with smoothed operator

$$P = \int_{\Omega} \nabla p \cdot u \; dV$$

Mechanical Power due to the pressure forces

$$\int_{\Omega} \nabla p \cdot u \; dV = -\int_{\Omega} p \; \mathrm{Div}(u) \; dV + \int_{\partial\Omega} p u \cdot n \; dS$$

$$\int_{\Omega} < \nabla p > \cdot u \; dV = -\int_{\Omega} p < \mathrm{Div}(u) > dV + \int_{\partial\Omega} p u \cdot n \; dS$$

unknown

Mechanical Power due to the pressure forces on the free surface

Smoothed differential operators

Energy conservation with smoothed operator

$$< \nabla p(r) >= \int_{\Omega} \left[p(r^*) + p(r) \right] \nabla W(r - r^*; h) \, dV^*$$

$$< \mathrm{Div}(u)(r) >= \int_{\Omega} \left[u(r^*) - u(r) \right] \bullet \nabla W(r - r^*; h) \, dV^*$$

$$\int_{\Omega} < \nabla p > \cdot u \, dV = -\int_{\Omega} p < \mathrm{Div}(u) > \, dV + \int_{\partial\Omega} p u \cdot n \, dS$$

=0

Using these smoothed operators the dynamic B.C. at the free surface, (for inviscid fluid) is satisfied in a weak sense.

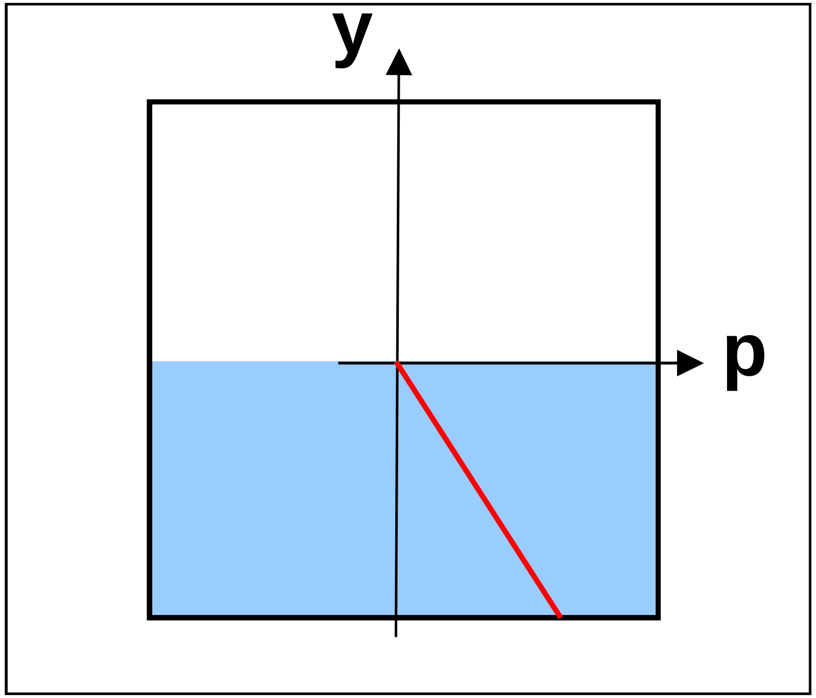
Colagrossi, M. Antuono, D. Le Touzé, **Theoretical considerations on the free-surface role in the smoothed-particle-hydrodynamics model**, *Physical Review E*, 79, 1-13, 2009.

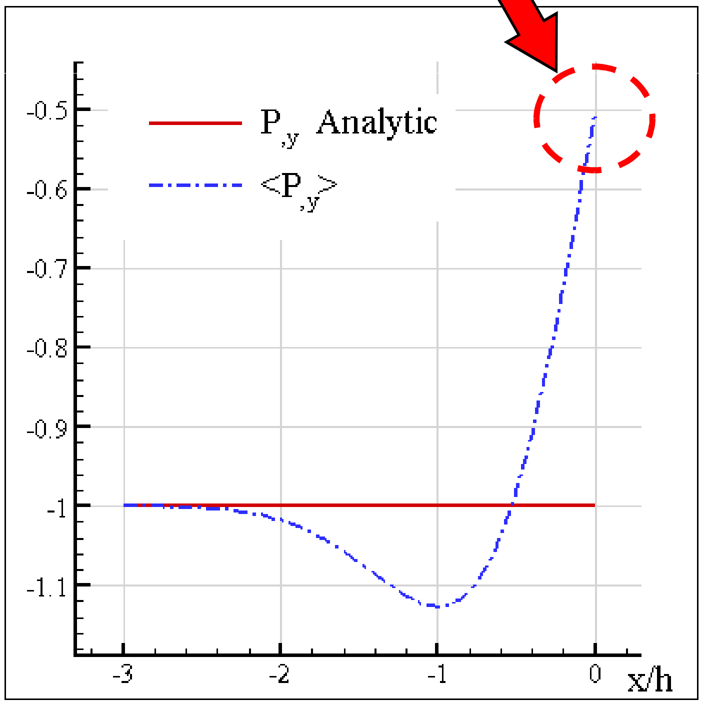
Smoothed differential operators

Smoothed pressure gradient consistency

$$<\nabla p(x)>=\int_{\Omega}\left[p(x^{*})+p(x)\right]\bullet\nabla W(x-x^{*};h)\;dV^{*}$$

Local inconsistency





Smoothed differential operators

Smoothed viscous terms

Monaghan & Gingold:

$$< \nabla \bullet V(r) >^{MG} = 2(d+2)\mu \int_{\Omega} \frac{\left[u(r^*) - u(r)\right] \bullet \left[r^* - r\right]}{\left|r^* - r\right|^2} \nabla W(r - r^*; h)\, dV^*$$

Galilean Invariance ☺

Morris:

$$< \nabla \bullet V(r) >^{Mo} = 2\mu \int_{\Omega} \frac{\left[r^* - r\right] \bullet \nabla W(r - r^*; h)}{\left|r^* - r\right|^2} \left[u(r^*) - u(r)\right]\, dV^*$$

Galilean Invariance ☹

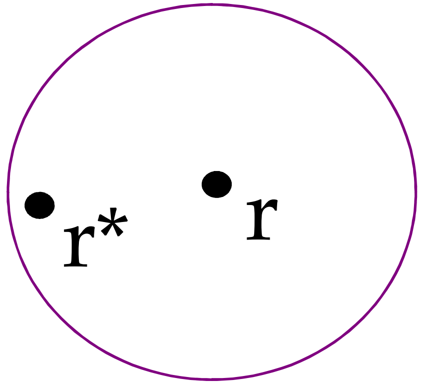
Smoothed viscous terms

Consistency inside the fluid domain

Taylor series inside the kernel support:

$$\left[u(r^*)-u(r)\right] =$$

$$\nabla u\big|_r (r^*-r)+\frac{1}{2}(r^*-r)\,H\big|_r\,(r^*-r)+O(\left|r^*-r\right|^3)$$



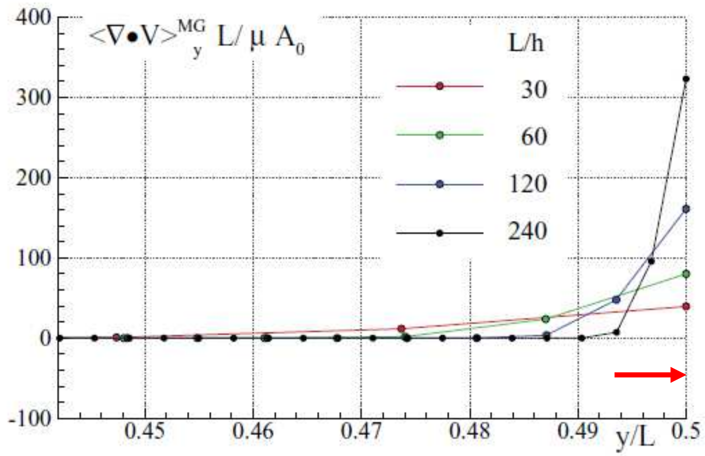
$$\lim_{h\to 0} <\nabla\bullet V>^{MG} = \mu\nabla^2 u + 2\mu\,\nabla div(u)$$

$$\lim_{h\to 0} <\nabla\bullet V>^{Mo} = \mu\nabla^2 u$$

Far from the free surface !!

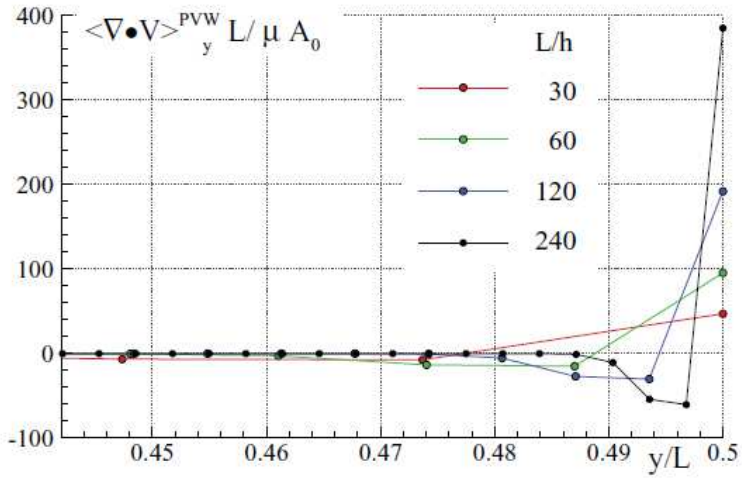
Smoothed viscous terms

Global consistency



O(1/h)

The viscous terms are singular at the free surface when *normal velocity gradients* exist.



$$\mathbb{V} = \lambda \,\mathrm{tr}\, \mathbb{D}\, \mathbb{1} + 2\,\mu\, \mathbb{D}.$$

$$\mathbb{L} = \left[\int_\Omega (r - r') \otimes \nabla W \, dV' \right]^{-1}$$

$$\langle \nabla \cdot \mathbb{V} \rangle^{\mathrm{PVW}} = \int_\Omega (\mathbb{L}'\, \mathbb{V}' + \mathbb{L}\, \mathbb{V}) \cdot \nabla W \, dV'.$$

J. Bonet and T. Lok, Comput. Methods Appl. Mech. Eng. 180, 97 (1999).

Smoothed viscous terms

Energy conservation for free-surface flow

$$P = \int_{\Omega} (\nabla \cdot V) \cdot u \; dV$$ Mechanical Power due to the viscous forces

$$\int_{\Omega} (\nabla \cdot V) \cdot u \; dV = -\int_{\Omega} V : D \; dV + \int_{\partial\Omega} Vn \cdot u \; dS$$

$$\int_{\Omega} < \nabla \cdot V >^{MG} \cdot u \; dV = -\int_{\Omega} V : D \; dV + \mu O(h)$$

$$\int_{\Omega} < \nabla \cdot V >^{Mo} \cdot u \; dV = -\mu \int_{\Omega} \left\| \nabla \mathrm{u} \right\|^2 \; dV + \mu O(h)$$

In the proximity of the Free surface

Smoothed differential operators

Global Consistency for the Mollified N.S. equation in the presence of free-surface

$$< \mathrm{Div}(u)(r) >= \int_{\Omega} \left[u(r^*) - u(r) \right] \cdot \nabla W(r - r^*; h)\, dV^*$$

$$< \nabla p(r) >= \int_{\Omega} \left[p(r^*) + p(r) \right] \nabla W(r - r^*; h)\, dV^*$$

$$< \nabla \cdot V(r) >^{MG} = 2(d+2)\mu \int_{\Omega} \frac{\left[u(r^*) - u(r) \right] \cdot \left[r^* - r \right]}{\left| r^* - r \right|^2} \nabla W(r - r^*; h)\, dV^*$$

A. Colagrossi, M. Antuono, A. Souto-Iglesias, D. Le Touzé, **Theoretical Analysis and numerical verification of the consistency of viscous SPH formulation in simulating free-surface flows**, *PHYSICAL REVIEW E* 84, 026705 (2011).

SPH scheme N°5

Derivation through discretization of the Mollified N.S. eq.

$$\begin{cases} \dfrac{D\rho_i}{Dt} = -\rho_i \displaystyle\sum_{j=1}^{N} (\mathrm{u}_j - \mathrm{u}_i)\cdot \nabla W_{ij} V_j ; \quad p_i = f(\rho_i) \\ \dfrac{D\mathrm{u}_i}{Dt} = -\dfrac{1}{\rho_i} \displaystyle\sum_{j=1}^{N} (p_i + p_j) \nabla W_{ij} V_j + F_i + \\ \qquad +2(d+2)\dfrac{\mu}{\rho_i} \displaystyle\sum_{j=1}^{N} \dfrac{(\mathrm{u}_j - \mathrm{u}_i)\cdot(\mathrm{x}_j - \mathrm{x}_i)}{\left\| \mathrm{x}_j - \mathrm{x}_i \right\|^2} \nabla W_{ij} V_j \\ \dfrac{D\mathrm{x}_i}{Dt} = \mathrm{u}_i \end{cases}$$

$$m_i = V_{0i} f^{-1}(p_{0i}) \qquad V_i(t) = \frac{m_i}{\rho_i(t)}$$

Free-surface flows

Multi-fluids flows

Conclusions

- Nowadays SPH is still a very powerful method for simulating violent free-surface flow.

- Five different SPH schemes have been derived following different theoretical approaches.

- When dealing with multi-fluids and free-surface flows only schemes 4 and 5 works.

- Global consistency and local inconsistency of Navier-Stokes SPH equations have been discussed.

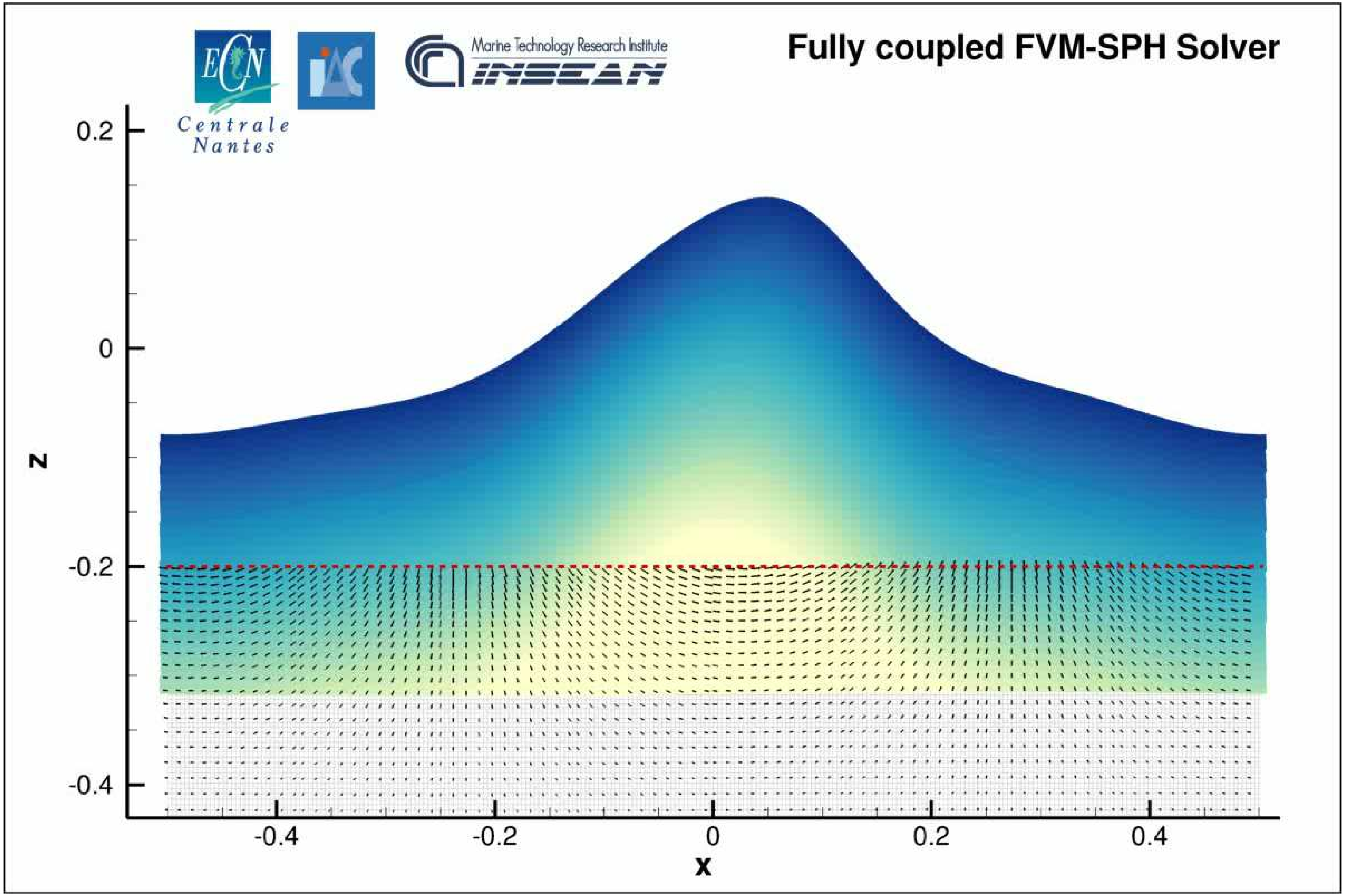
Multi-purpose interfaces for coupling SPH with other solvers

Spheric 2013, (Trondheim, Norway)

- SPH used only in regions close to the free surface
- FV incompressible solver, implicit scheme with level-set approach
- Dynamic overlapping grids (CHIMERA) to force the solution coming from SPH

B. Bouscasse, S. Marrone, A. Colagrossi, A. Di Mascio., *Multi-purpose interfaces for coupling SPH with other solvers*, 8th SPHERIC conference, Trondheim Norway (2013)

Multi-purpose interfaces for coupling SPH with other solvers



Useful Links

- CNR-INSEAN website http://www.insean.cnr.it/
- SPH INSEAN Youtube Channel https://www.youtube.com/channel/UCgxrxWzZi61095v2Zj73KTw
- http://www.insean.cnr.it/content/COLAGROSSI-ANDREA
- http://scholar.google.it/citations?user=itNJWkEAAAAJ&hl=it
- SPHERIC website (SPHERIC - SPH European Research Interest Community)

The End

Thanks for your attention