The finite volume particle method for flows with moving walls

Nathan J. Quinlan

Mechanical and Biomedical Engineering
National University of Ireland, Galway

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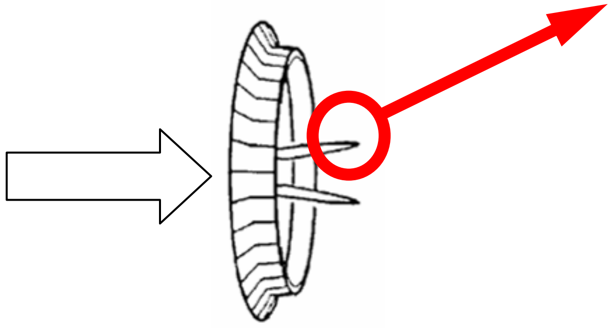
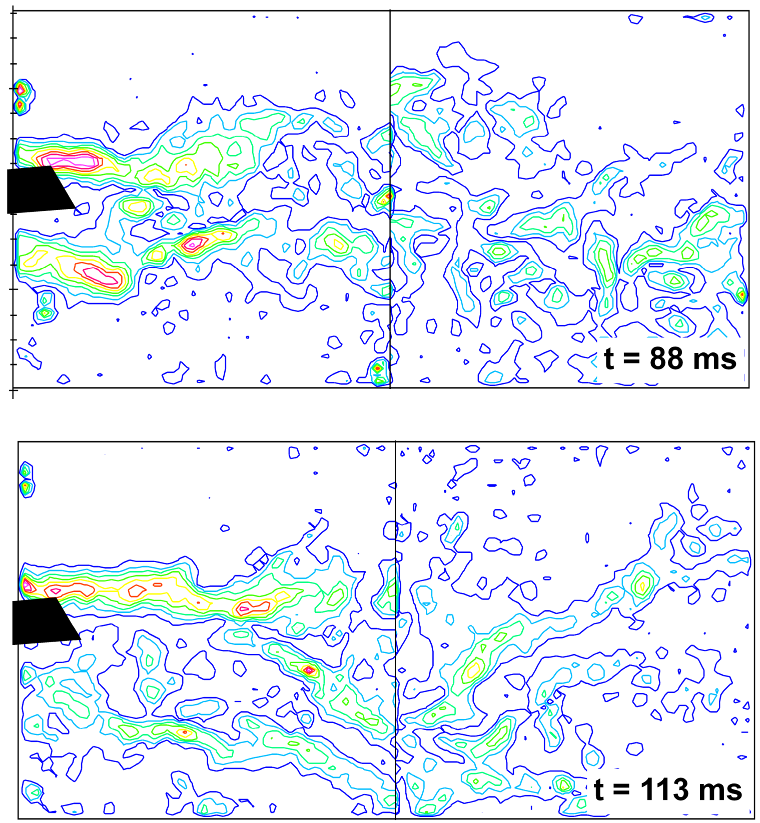
OÉ Gaillimh

5th SPHERIC workshop
Manchester, 2010

Motivation: biomedical fluid dynamics





Bellofiore *et al.*, 2010

Mechanical
heart valve

Re $\cong$ 6000

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The lineage of FVPM

Hietel, Steiner, Struckmeier	2000	***A finite-volume particle method for compressible flows*** 2D, 1st order
Junk	2001	***Do finite volume methods need a mesh?***
Ismagilov	2005	Smooth volume integral method 1D with MUSCL
Keck, Hietel	2005	Incompressible flow
Nestor *et al.*	2008	2D with MUSCL, viscous flows
Nestor, Quinlan	2009	Incompressible, moving body

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The finite volume particle method

Conservation law: $$\frac{\partial \mathbf{U}}{\partial t} + \nabla \bullet \mathbf{F}(\mathbf{U}) = 0$$

Introduce a **compactly supported** test function $\psi_i(\mathbf{x})$:

weak form: $$\int_\Omega \psi_i \frac{\partial \mathbf{U}}{\partial t} \mathrm{d}\mathbf{x} + \int_\Omega \psi_i \nabla \bullet \mathbf{F}(\mathbf{U}) \mathrm{d}\mathbf{x} = 0$$

$$\int_\Omega \psi_i \frac{\partial \mathbf{U}}{\partial t} d\mathbf{x} - \int_\Omega \nabla \psi_i \bullet \mathbf{F}(\mathbf{U}) d\mathbf{x} = 0$$

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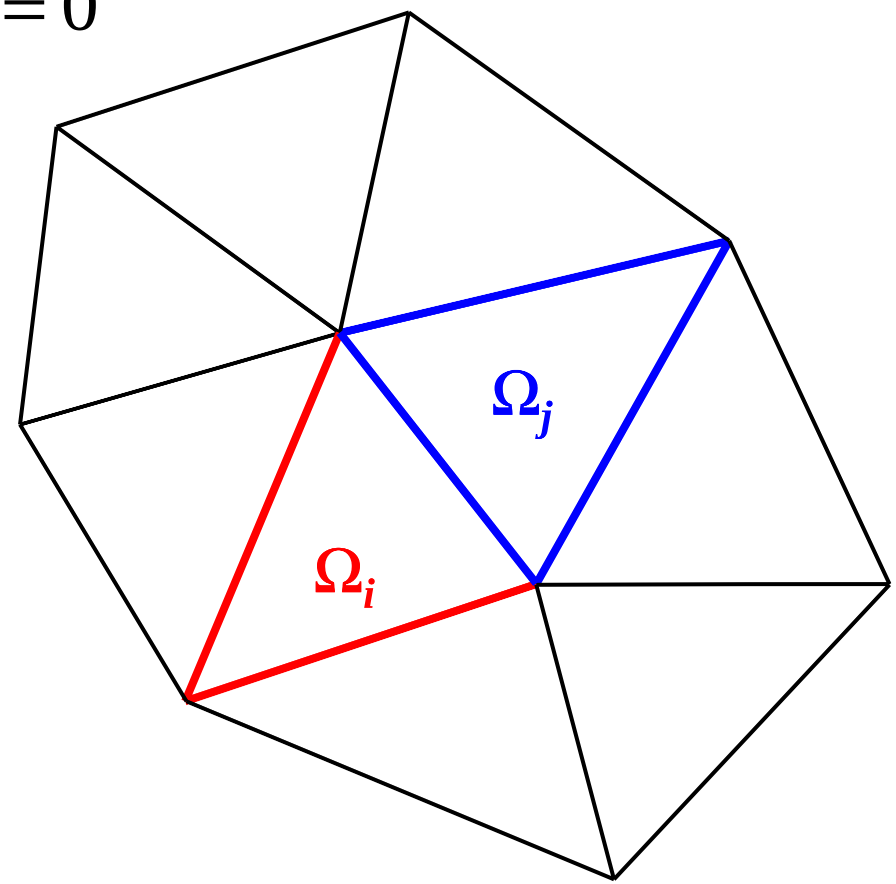
Choice of test function and support volume

$$\int_{\Omega} \psi_i \frac{\partial \mathbf{U}}{\partial t} d\mathbf{x} - \int_{\Omega} \nabla \psi_i \bullet \mathbf{F}(\mathbf{U}) d\mathbf{x} = 0$$



$$\psi_i(\mathbf{x}) = \begin{cases} 1 & x \in \Omega_i \\ 0 & \text{otherwise} \end{cases}$$

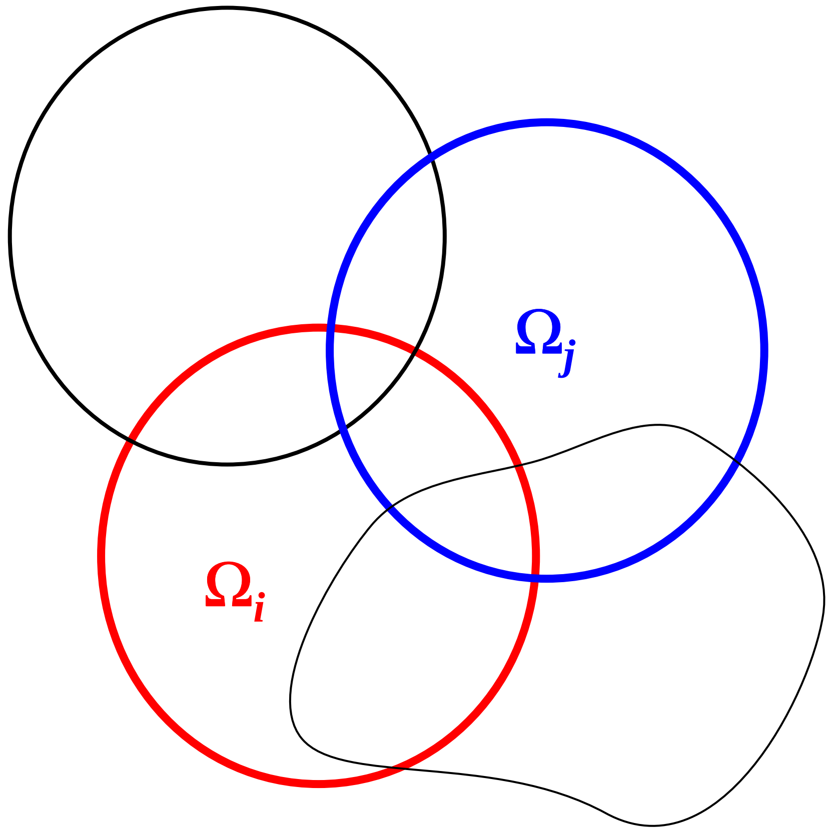
→ finite volume method

Choice of test function and support volume

$$\int_{\Omega} \psi_i \frac{\partial \mathbf{U}}{\partial t} \mathrm{d}\mathbf{x} - \int_{\Omega} \nabla \psi_i \bullet \mathbf{F}(\mathbf{U}) \mathrm{d}\mathbf{x} = 0$$

$$\psi_i(\mathbf{x}) = \frac{W_i(\mathbf{x})}{\sum_k W_k(\mathbf{x})}$$

where $W_i(\mathbf{x}) = 0$ for $x \notin \Omega_i$



FVPM

→ finite volume ***particle*** method

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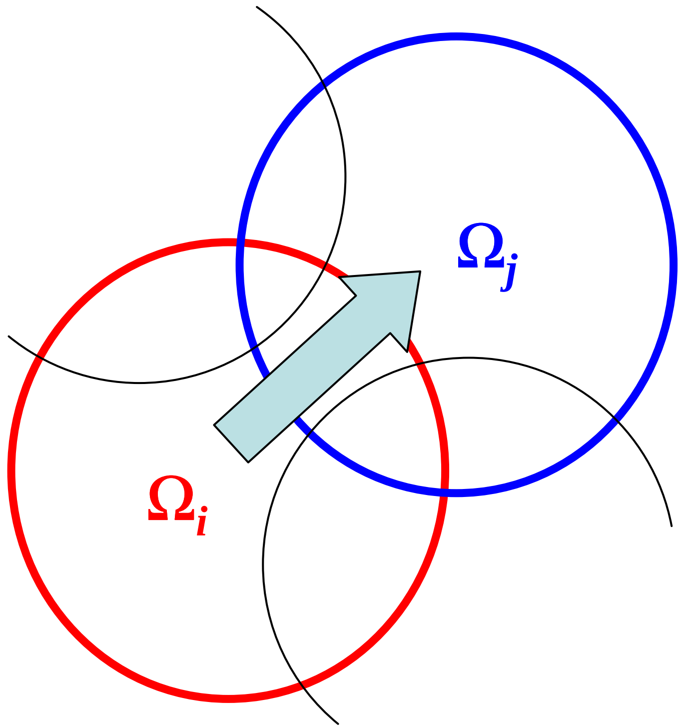
Interpretation in terms of pair interactions

$$\int_\Omega \psi_i \frac{\partial \mathbf{U}}{\partial t} \mathrm{d}\mathbf{x} - \int_\Omega \nabla\psi_i \bullet \mathbf{F}(\mathbf{U}) \mathrm{d}\mathbf{x} = 0$$

$$\sum_j \frac{W_i(\mathbf{x}) \nabla W_j(\mathbf{x}) - W_j(\mathbf{x}) \nabla W_i(\mathbf{x})}{\left(\sum_k W_k(\mathbf{x})\right)^2}$$



$$\int_\Omega \psi_i \frac{\partial \mathbf{U}}{\partial t} \mathrm{d}\mathbf{x} - \sum_j \int_\Omega \left(\gamma_{ij} - \gamma_{ji}\right) \bullet \mathbf{F}(\mathbf{U}) \mathrm{d}\mathbf{x} = 0$$

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3 approximations in FVPM, as in finite volume

$$\frac{d}{dt}\int_\Omega \psi_i \mathbf{U} d\mathbf{x} - \sum_j \int_\Omega \left(\gamma_{ij} - \gamma_{ji}\right) \bullet \mathbf{F}(\mathbf{U}) d\mathbf{x} - \int_\Omega \frac{\partial \psi_i}{\partial t} \mathbf{U} d\mathbf{x} = 0$$

(1) Replace the weighted volume average of **U** with a "particle" value

(2) Represent **F**(**U**(**x**,t)) with a single value for the overlap region

$$\frac{d}{dt}\left(V_i \mathbf{U}_i\right) - \sum_j \beta_{ij} \mathbf{F}_{ij} - \int_\Omega \frac{d\psi_i}{dt} \mathbf{U} d\mathbf{x} = 0$$

where

$$V_i = \int_\Omega \psi_i d\mathbf{x}$$

(3) Reconstruct $\mathbf{U}_i$, $\mathbf{U}_j$ at interface for Riemann problem

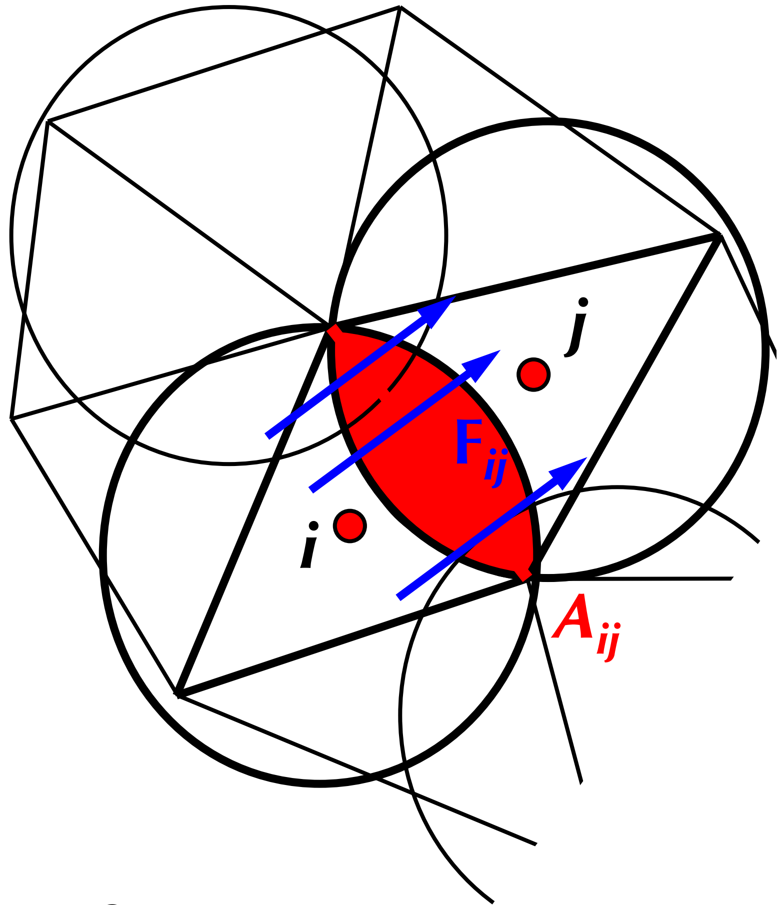
$$\rightarrow \mathbf{F}_{ij} \cong \mathbf{F}\left(\mathbf{U}_i, \mathbf{U}_j\right)$$

Analogy with mesh finite volume method

FVM

$$\frac{\mathrm{d}}{\mathrm{d}t}(V_i\mathbf{U}_i)-\sum_j \mathbf{A}_{ij}\bullet\left(\mathbf{F}_{ij}+\dot{\mathbf{x}}_{ij}\mathbf{U}_{ij}\right)=0$$

A_{ij} ⟷ β_{ij}



FVPM

$$\frac{\mathrm{d}}{\mathrm{d}t}(V_i\mathbf{U}_i) - \sum_j \beta_{ij}\bullet\left(\mathbf{F}_{ij}+\dot{\mathbf{x}}_{ij}\mathbf{U}_{ij}\right) = 0$$

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The particle interaction vector

$$\beta_{ij} = \int \frac{W_i \nabla W_j - W_j \nabla W_i}{\left(\sum_k W_k(\mathbf{x})\right)^2} \mathrm{d}\,\mathbf{x}$$

W(x) is any kernel with compact support

2 properties of β_{ij}

$\beta_{ij} = -\beta_{ij}$ symmetry $\Rightarrow$ **exact conservation**

$\sum_j \beta_{ij} = 0$ the particle volume is "closed" $\Rightarrow$ **zero-order consistency**

The mesh finite volume method is a special case of FVPM.

(Junk, 2003)

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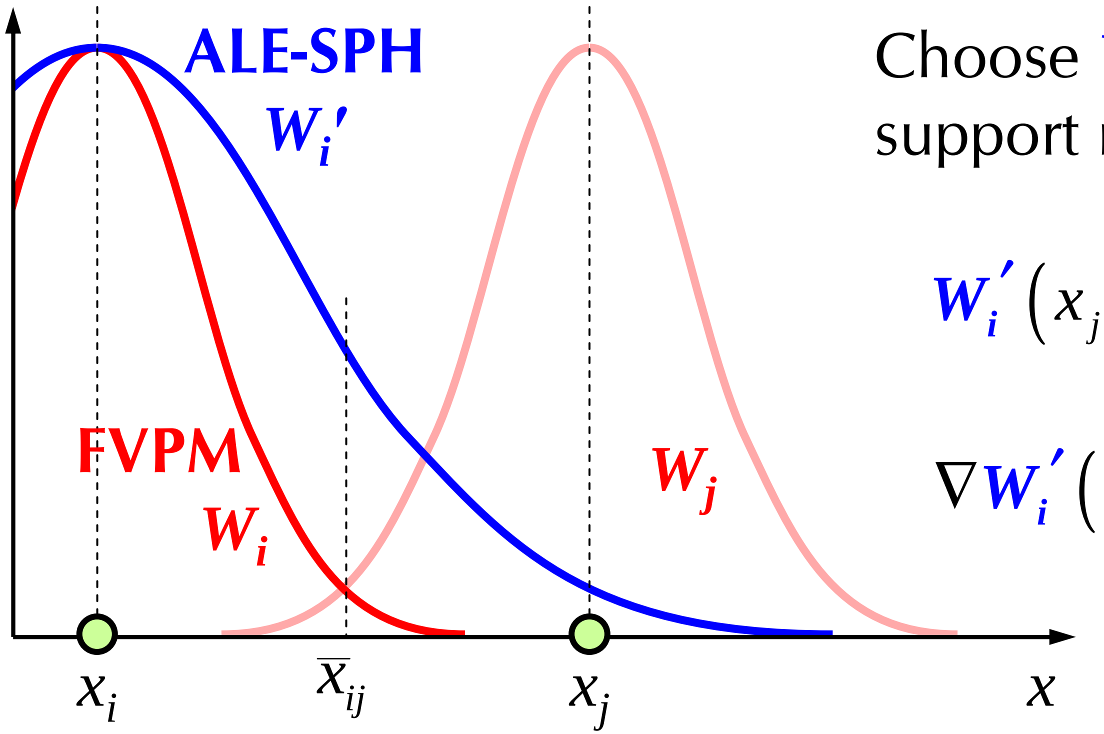
Relationship to ALE-SPH

$$\frac{\mathrm{d}}{\mathrm{d}t}(V_i\mathbf{U}_i) - 2V_i\sum_j V_j\mathbf{G}\left(\mathbf{U}_i,\mathbf{U}_j\right)\bullet\nabla W_i'\left(\mathbf{x}_j\right) = 0$$

Vila (1999)



Choose W' with double the support radius of W ⇒

$$W_i'\left(x_j\right) = W_i\left(\bar{x}_{ij}\right)$$

$$\nabla W_i'\left(x_j\right) = \frac{1}{2}\nabla W_i\left(\bar{x}_{ij}\right)$$

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Relationship to ALE-SPH

Shepard-normalised RSPH kernel: $\tilde{W}_i'(\mathbf{x}) = \dfrac{W_i'(\mathbf{x})}{\sum_k W_k'(\mathbf{x}) V_k}$

Approximate relationship: $\nabla \tilde{W}_i'(x_j) \cong \frac{1}{2} \nabla \tilde{W}_i(x_{ij})$

$$\nabla \tilde{W}_i'(x_j) = \sum_j \left[\frac{W_i' \nabla W_j' - W_j' \nabla W_i'}{\left(\sum_k W_k' \right)^2 V} \right]_{x=x_j} \cong \frac{1}{2} \sum_j \left[\frac{W_i \nabla W_j - W_j \nabla W_i}{\left(\sum_k W_k \right)^2 V} \right]_{x=\bar{x}_{ij}}$$

(if $V_i = V_j = V$)

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Relationship to ALE-SPH

ALE-SPH approximates

$$\frac{\mathrm{d}}{\mathrm{d}t}(V_i\mathbf{U}_i) - \sum_j \mathbf{G}(\mathbf{U}_i,\mathbf{U}_j)\bullet V_i\left[\frac{W_i\nabla W_j - W_j\nabla W_i}{\left(\sum_k W_k\right)^2}\right]_{x=\bar{x}_{ij}} = 0$$

FVPM is

$$\frac{\mathrm{d}}{\mathrm{d}t}(V_i\mathbf{U}_i) - \sum_j \mathbf{G}(\mathbf{U}_i,\mathbf{U}_j)\bullet \int_{\Omega_i\cap\Omega_j} \frac{W_i\nabla W_j - W_j\nabla W_i}{\left(\sum_k W_k\right)^2}\,\mathrm{d}\mathbf{x} = 0$$

overlap volume ≅ material volume

⇒ **RSPH ≅ FVPM with a single-point approximation to β_{ij}**

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A continuum from SPH to finite volume?

SPH $$-\left(\frac{p_i}{\rho_i^2}+\frac{p_j}{\rho_j^2}\right)\nabla W_{ij} V_j$$

ALE-SPH $$-2V_i V_j \nabla W_{ij} \mathbf{F}\left(\mathbf{U}_i, \mathbf{U}_j\right)$$

FVPM $$-\boldsymbol{\beta}_{ij} \mathbf{F}\left(\mathbf{U}_i, \mathbf{U}_j\right)$$

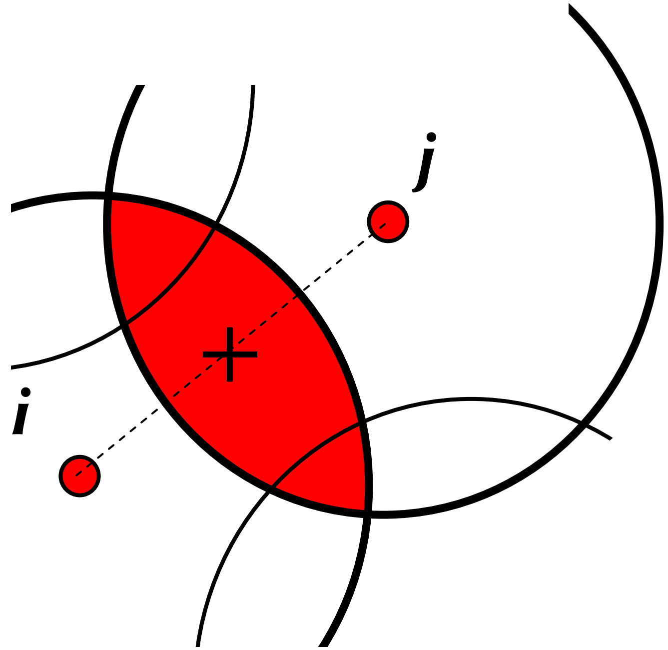
finite volume $$-\mathbf{A}_{ij} \mathbf{F}\left(\mathbf{U}_i, \mathbf{U}_j\right)$$

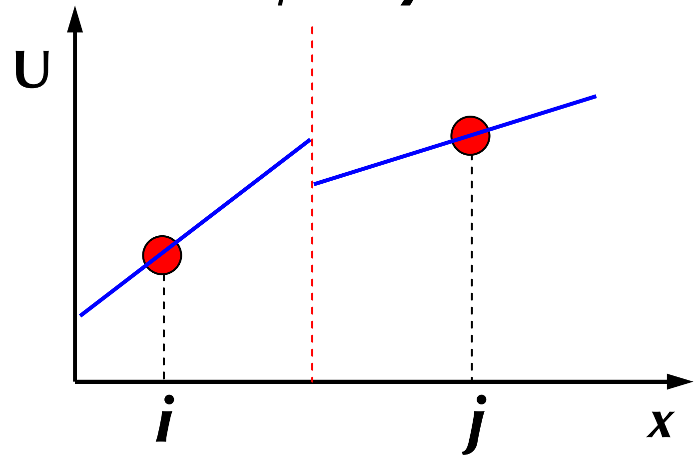
Higher-order spatial accuracy by MUSCL

- Evaluate gradients at particle barycentres using (corrected) SPH approximation

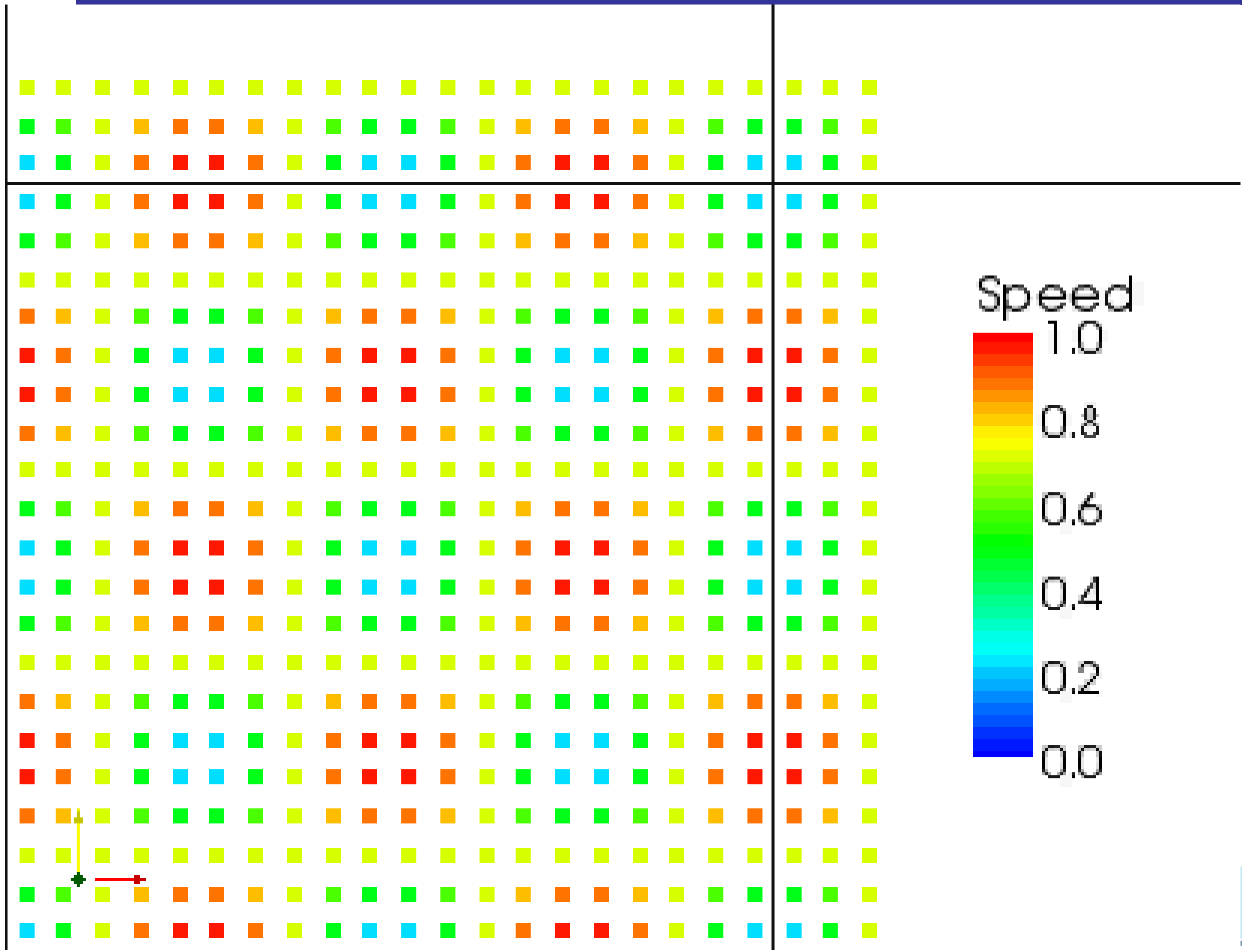
- Reconstruct $\mathbf{U}_L$ and $\mathbf{U}_R$ on both sides of interface

- Compute approximate numerical flux $F(\mathbf{U}_L, \mathbf{U}_R)$





Taylor-Green flow at Re = 100, Lagrangian



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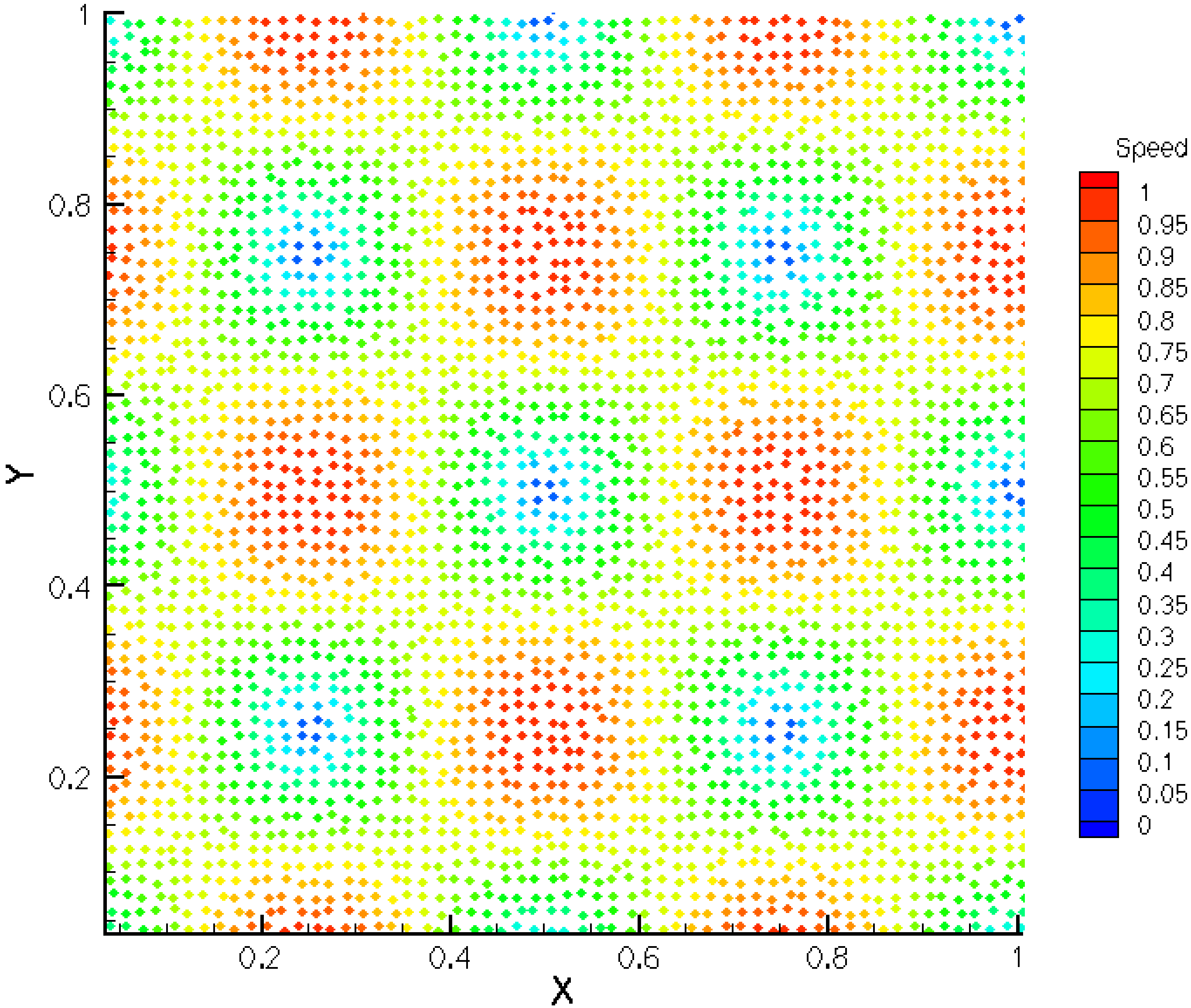
Randomised initialsiation, Lagrangian



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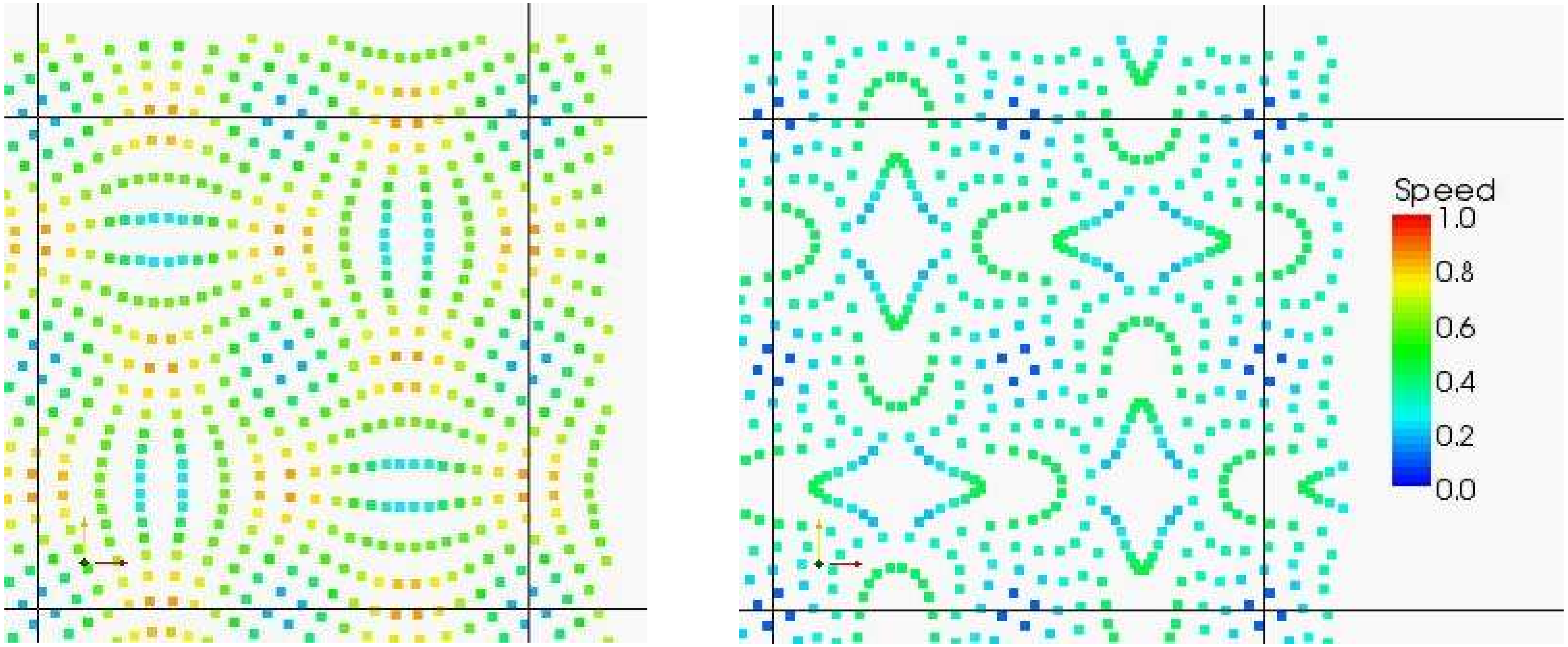
Taylor-Green, Re = 100, corrected Lagrangian



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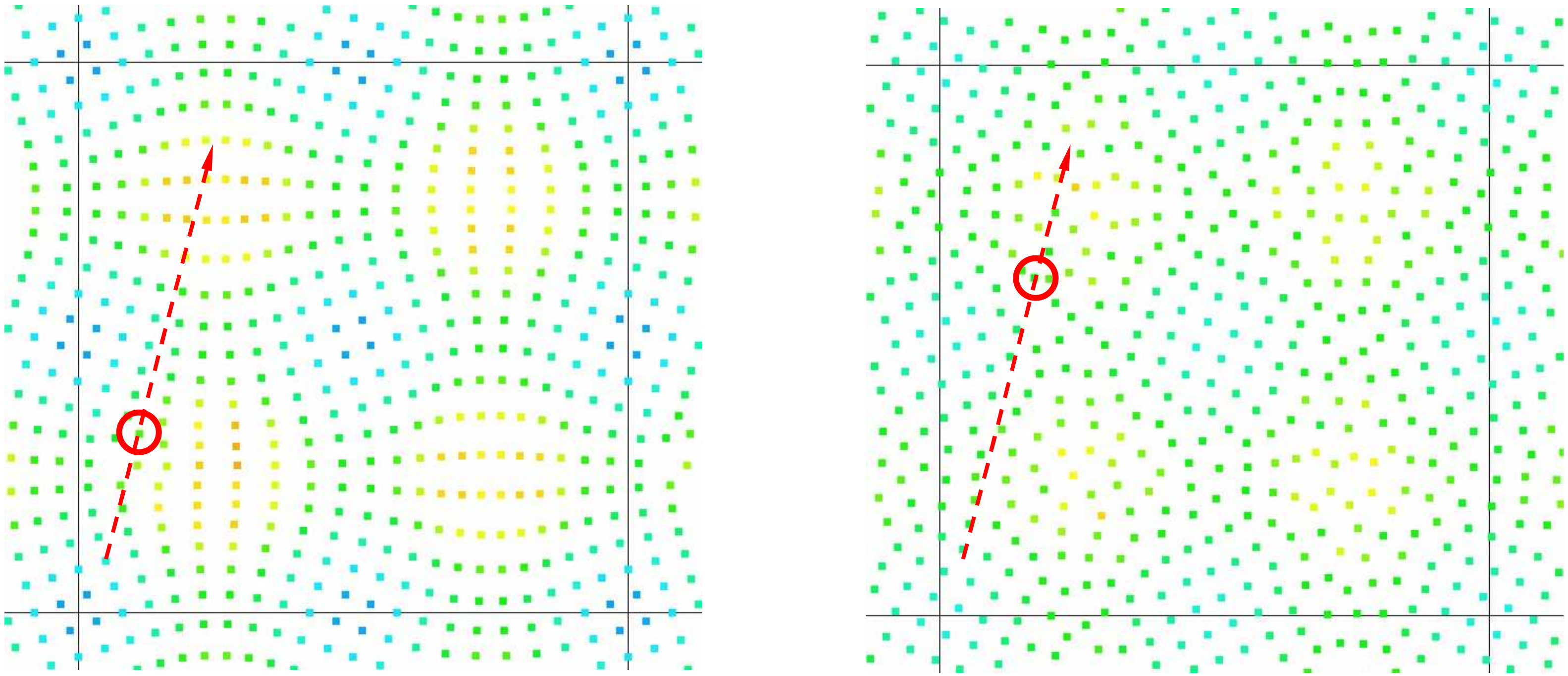
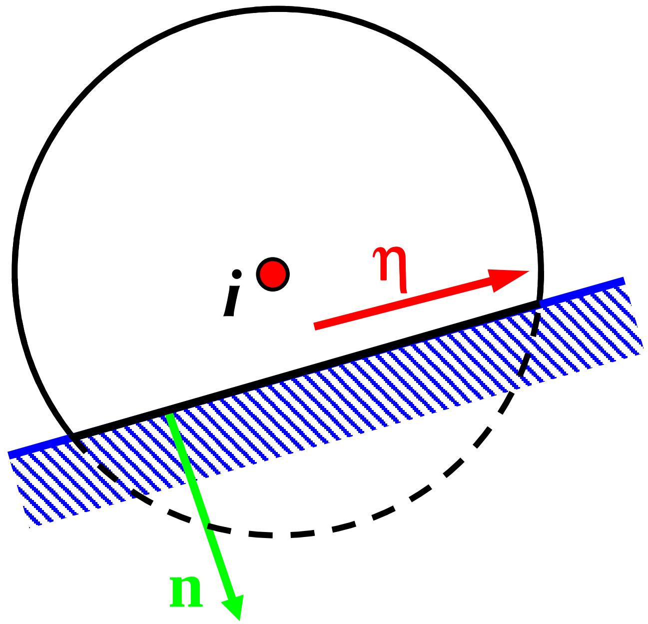
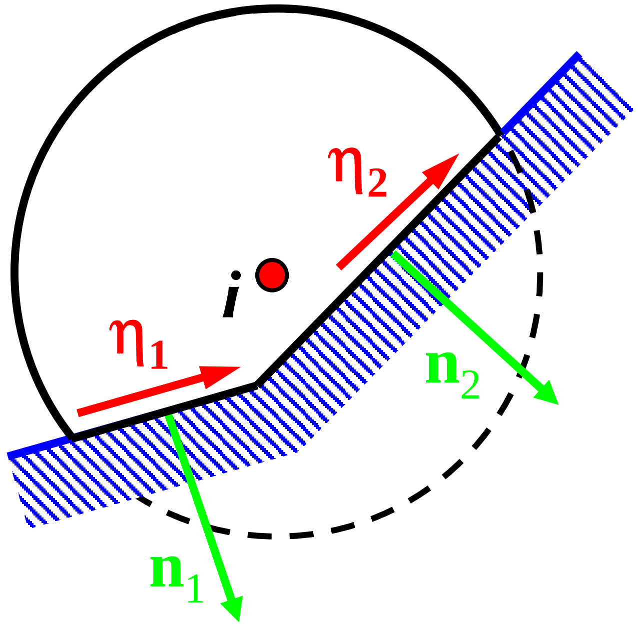
Taylor-Green flow with rogue particle



Boundary conditions

Particle support is truncated at boundary.





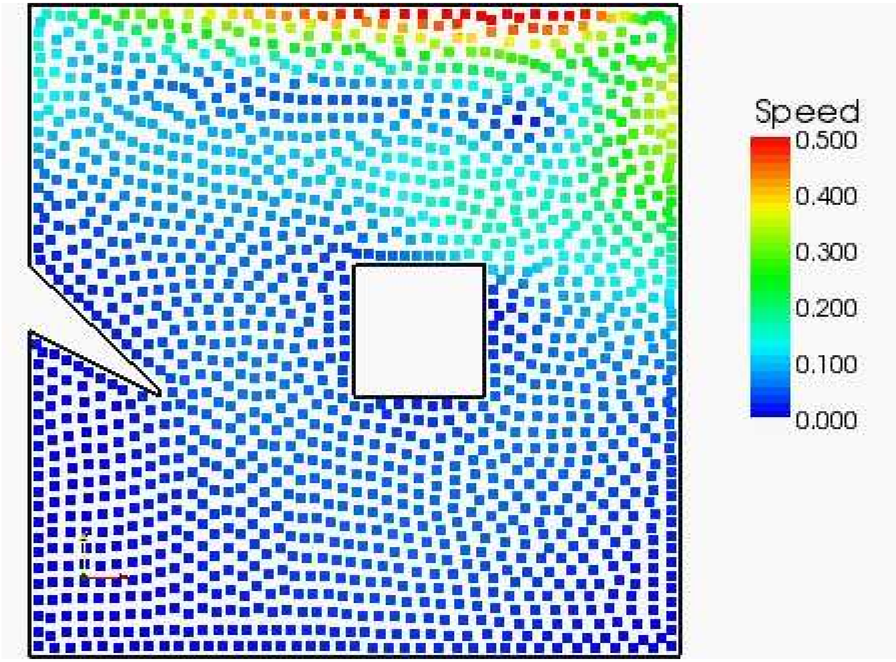
Compute boundary interaction vector directly…

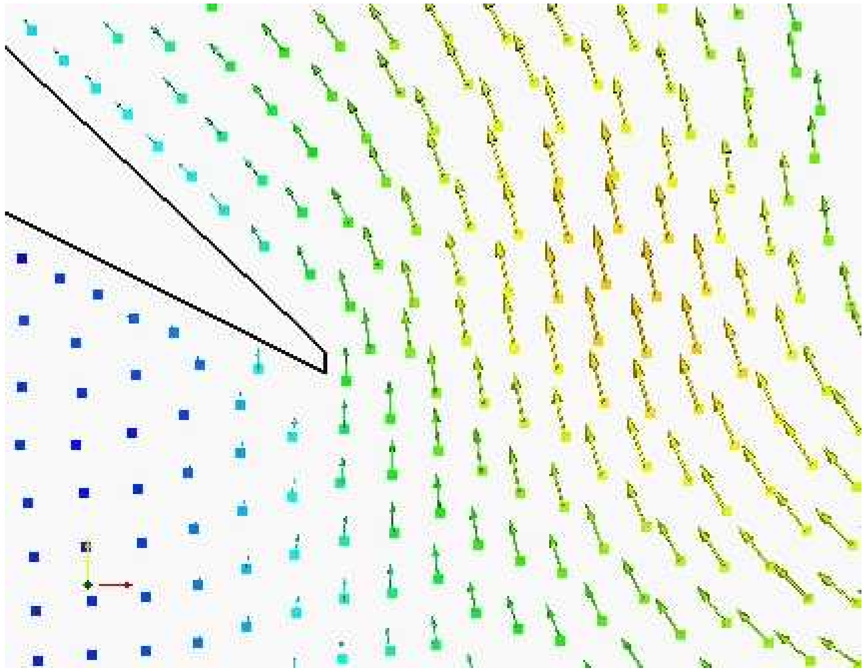
$$\beta_i^b = \int \frac{W_i}{\sum_k W_k(\mathbf{x})} \mathbf{n} \, d\eta$$

…or by enforcing

$$\sum_j \beta_{ij} + \beta_i^b = 0$$

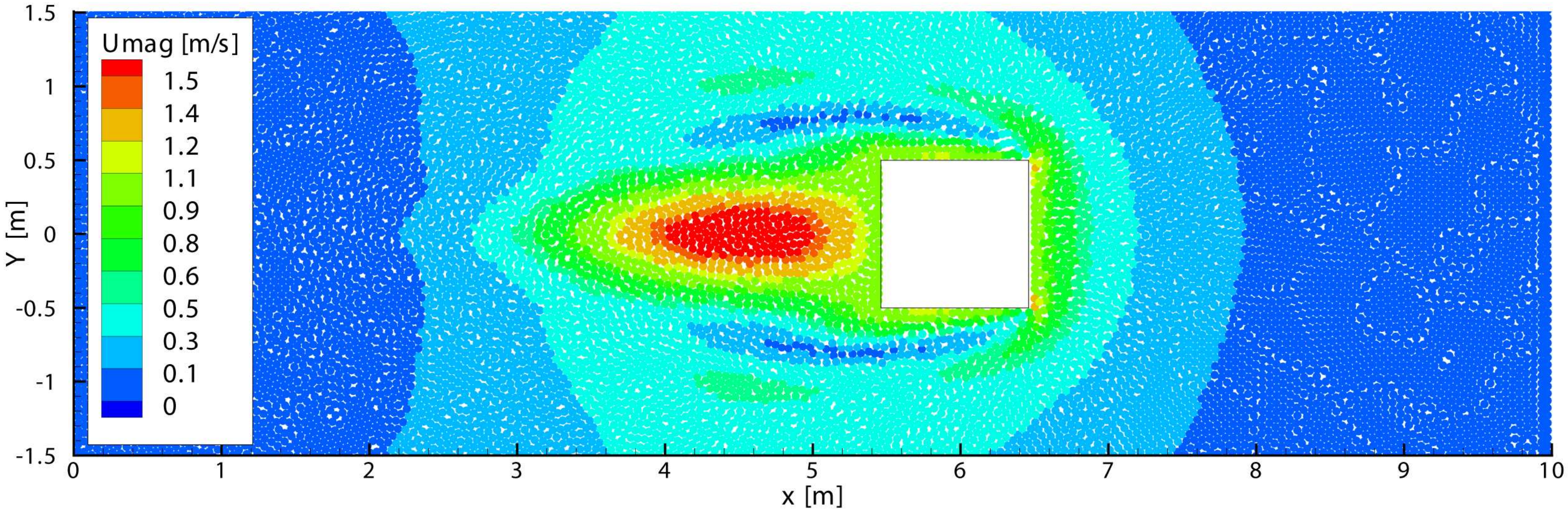
“Complex” geometry – Re_L=100



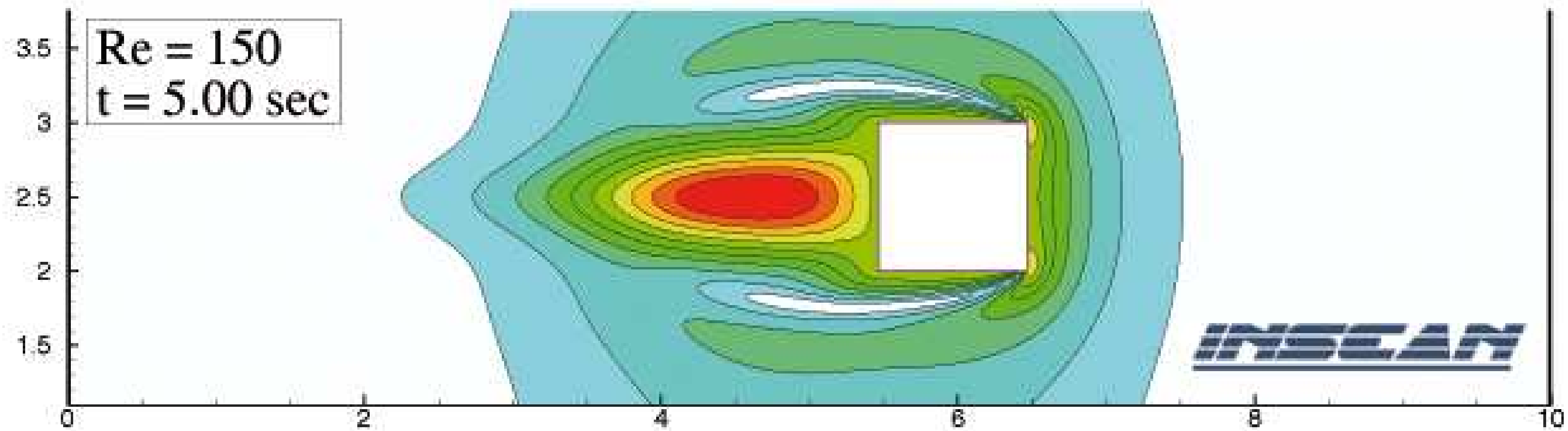


SPHERIC benchmark 6: moving square

FVPM



Level set (Colagrossi)



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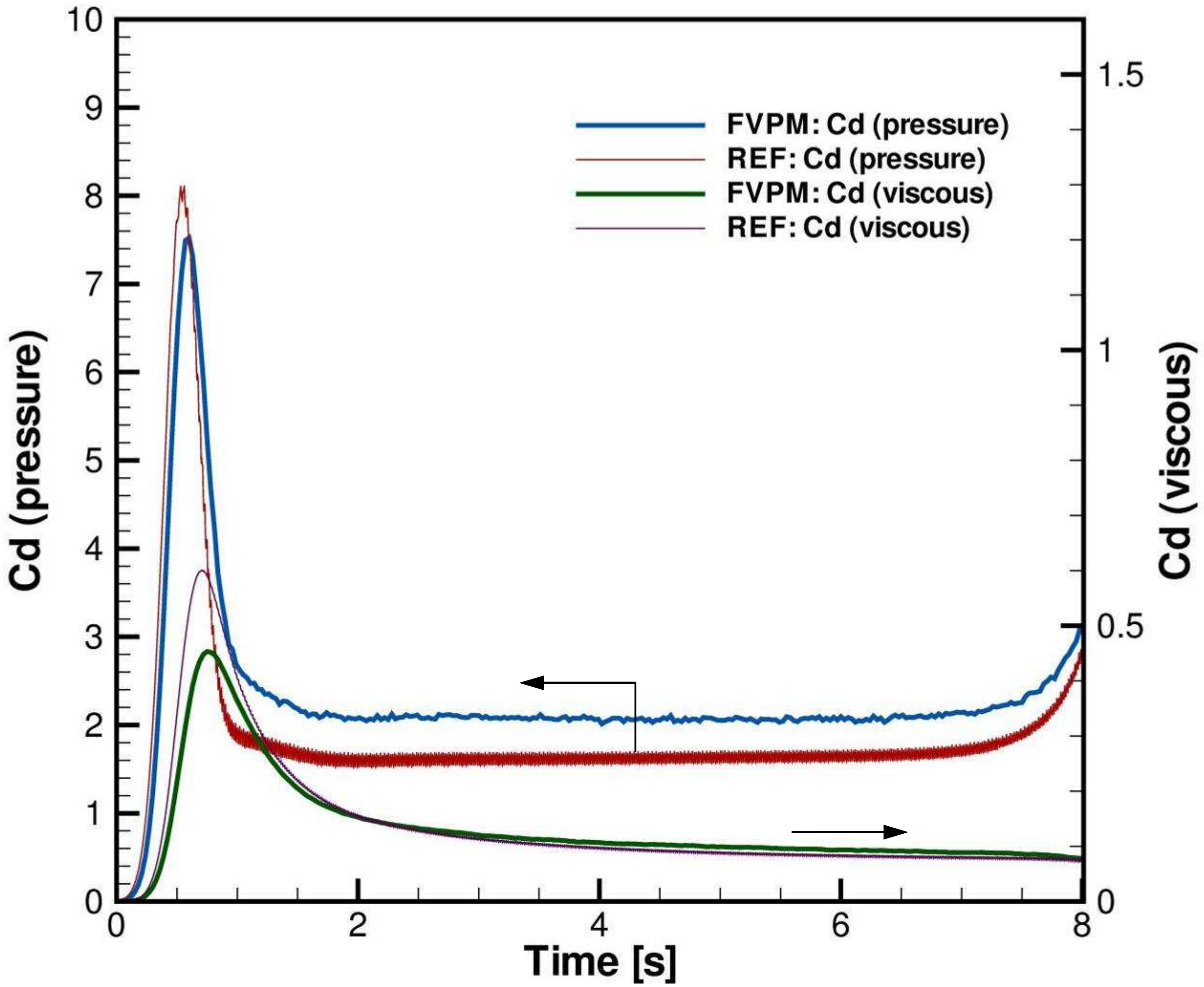
SPHERIC benchmark 6: moving square



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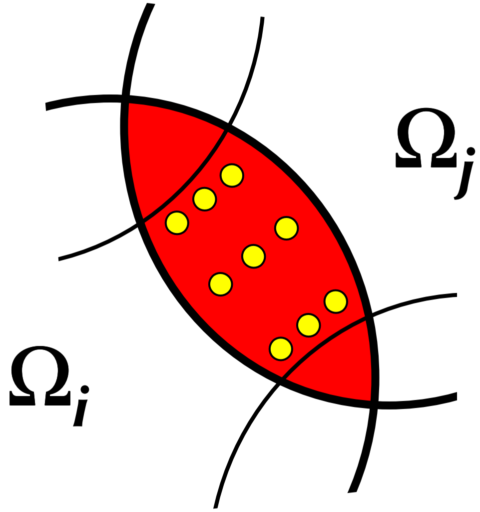
Correction of numerical β_{ij}

$$\beta_{ij} = \int \frac{W_i \nabla W_j - W_j \nabla W_i}{\left(\sum_k W_k(\mathbf{x}) \right)^2} \mathrm{d}\mathbf{x}$$



Numerical integration is necessary.

Typically 6×6 quadrature points.

Correction options

Self-flux (Teleaga and Struckmeier, 2008)

- Preserves uniform states
- violates conservation

Pairwise shifting (Hietel and Keck, 2003)

- Restores conservation
- Errors are shifted to neighbouring particles

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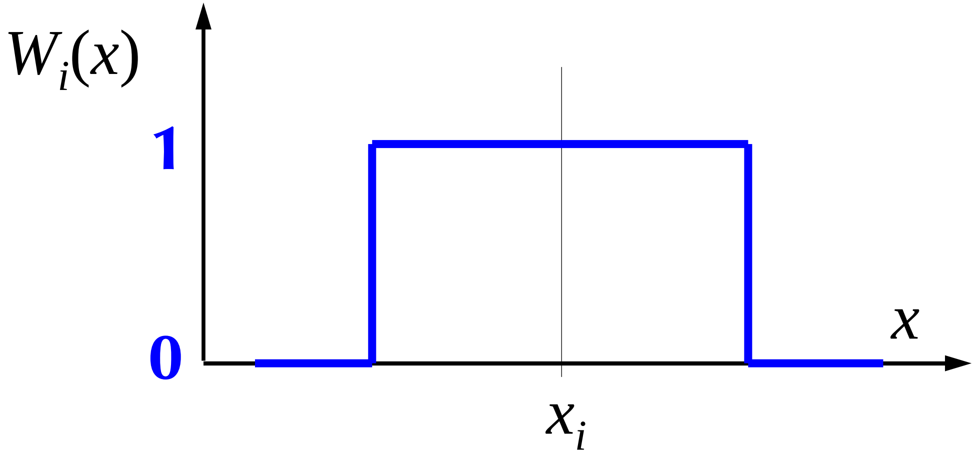
Computation time

neighbour search	$<1\%$
flux	4%
gradients	2%
particle update	2%
motion correction	$<1\%$
β_{ij}	74%
barycentres	14%

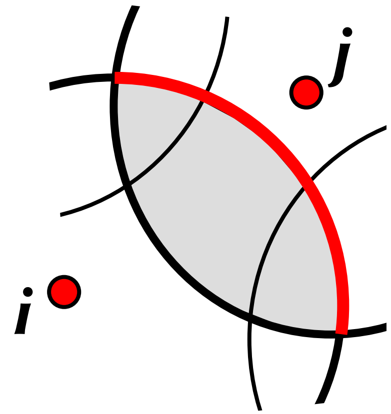
Exact (and fast) evaluation of β_{ij}

Choose the simplest possible kernel



$$W_i(x) = \begin{cases} 1 & x \in \Omega_i \\ 0 & \text{otherwise} \end{cases}$$

$$\beta_{ij} = \int \frac{W_i \nabla W_j - W_j \nabla W_i}{\left(\sum_k W_k(\mathbf{x})\right)^2} \mathrm{d}\,\mathbf{x}$$



$\nabla W_i = 0$ everywhere except on boundary of *i*

Integration over $\Omega_i \cap \Omega_j$ reduces to integration along a curve

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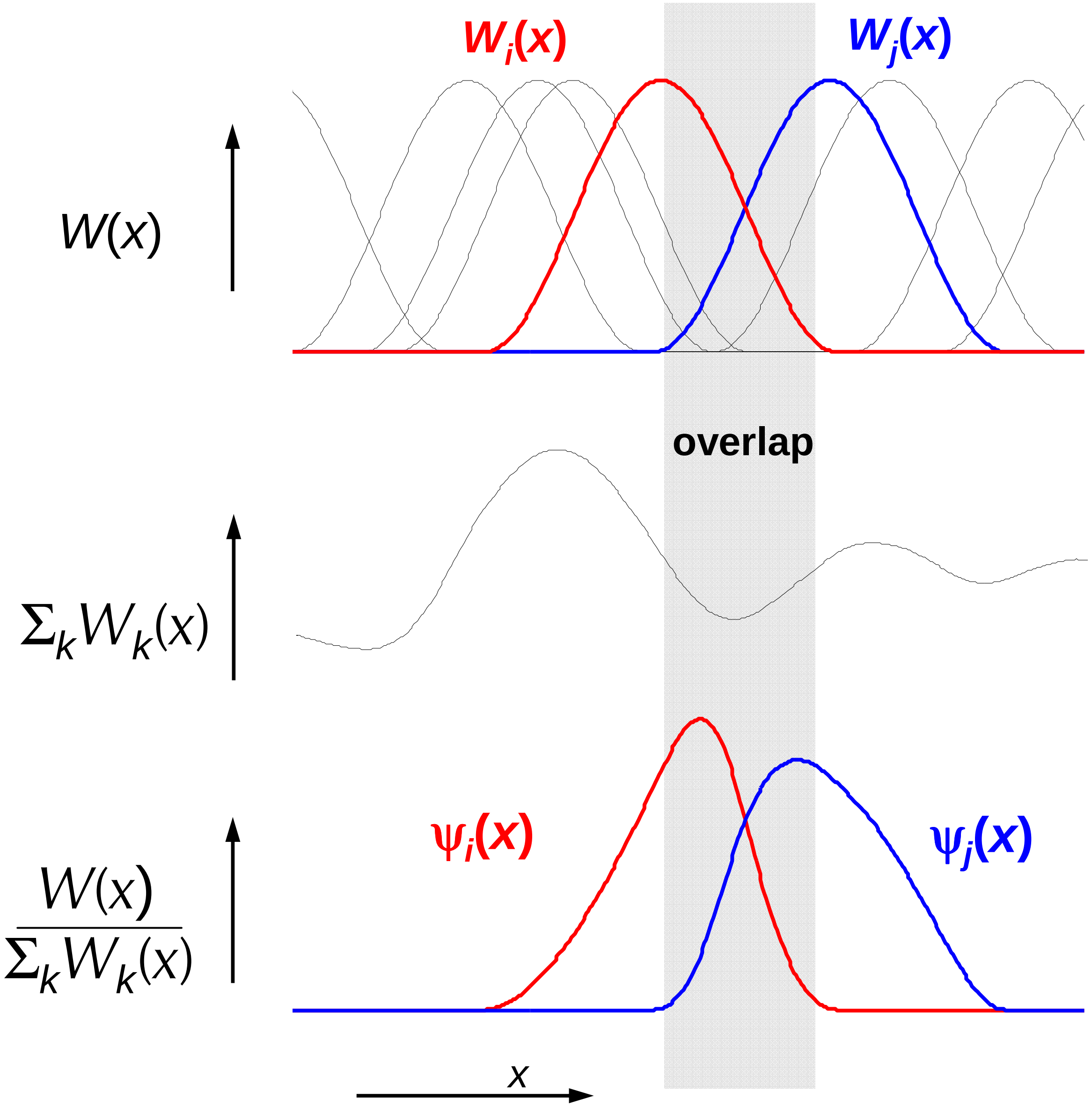
Smooth kernel functions



$$\beta_{ij} = \int \frac{W_i \nabla W_j - W_j \nabla W_i}{\left(\sum_k W_k(\mathbf{x}) \right)^2} \, \mathrm{d}\mathbf{x}$$

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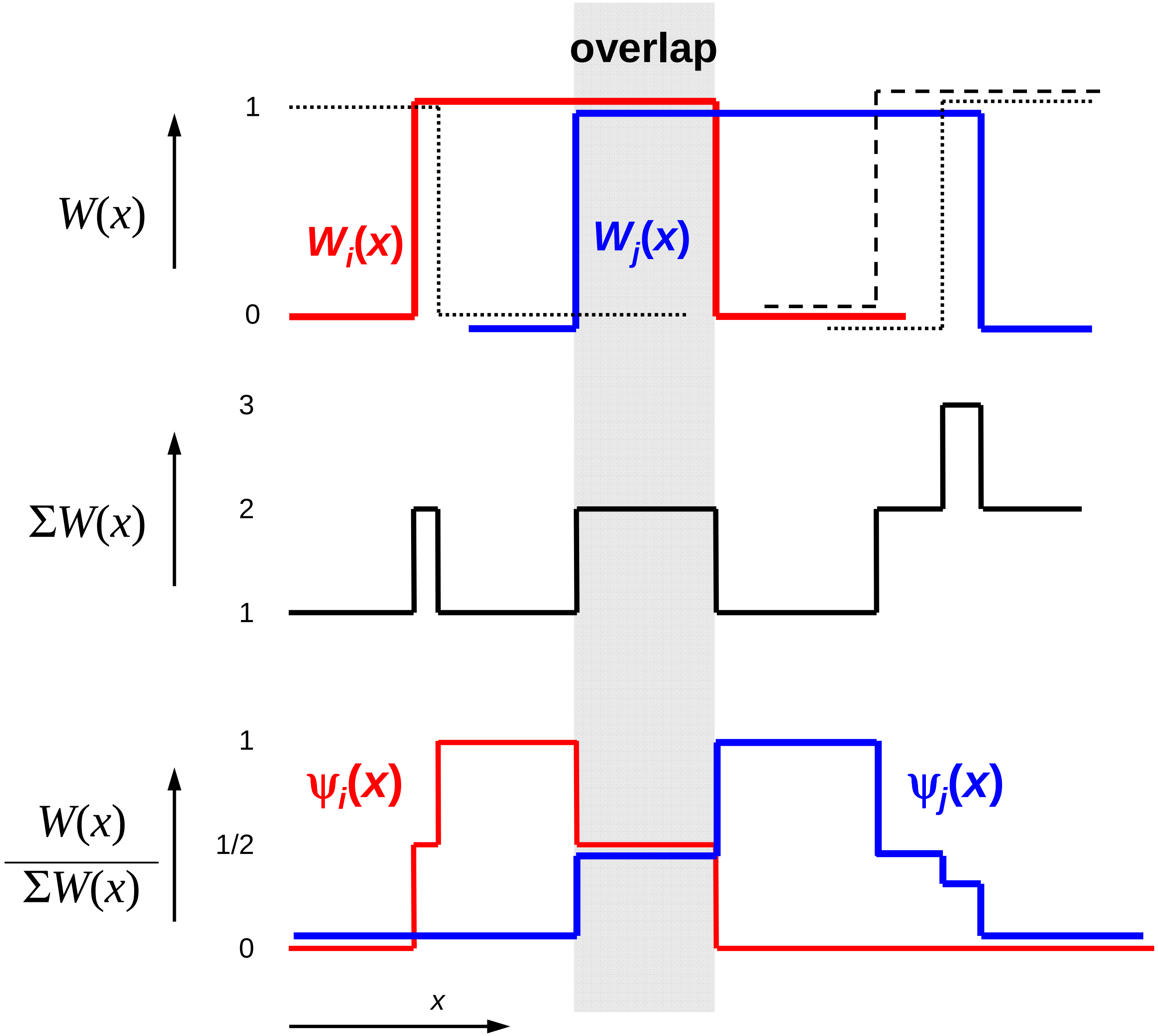
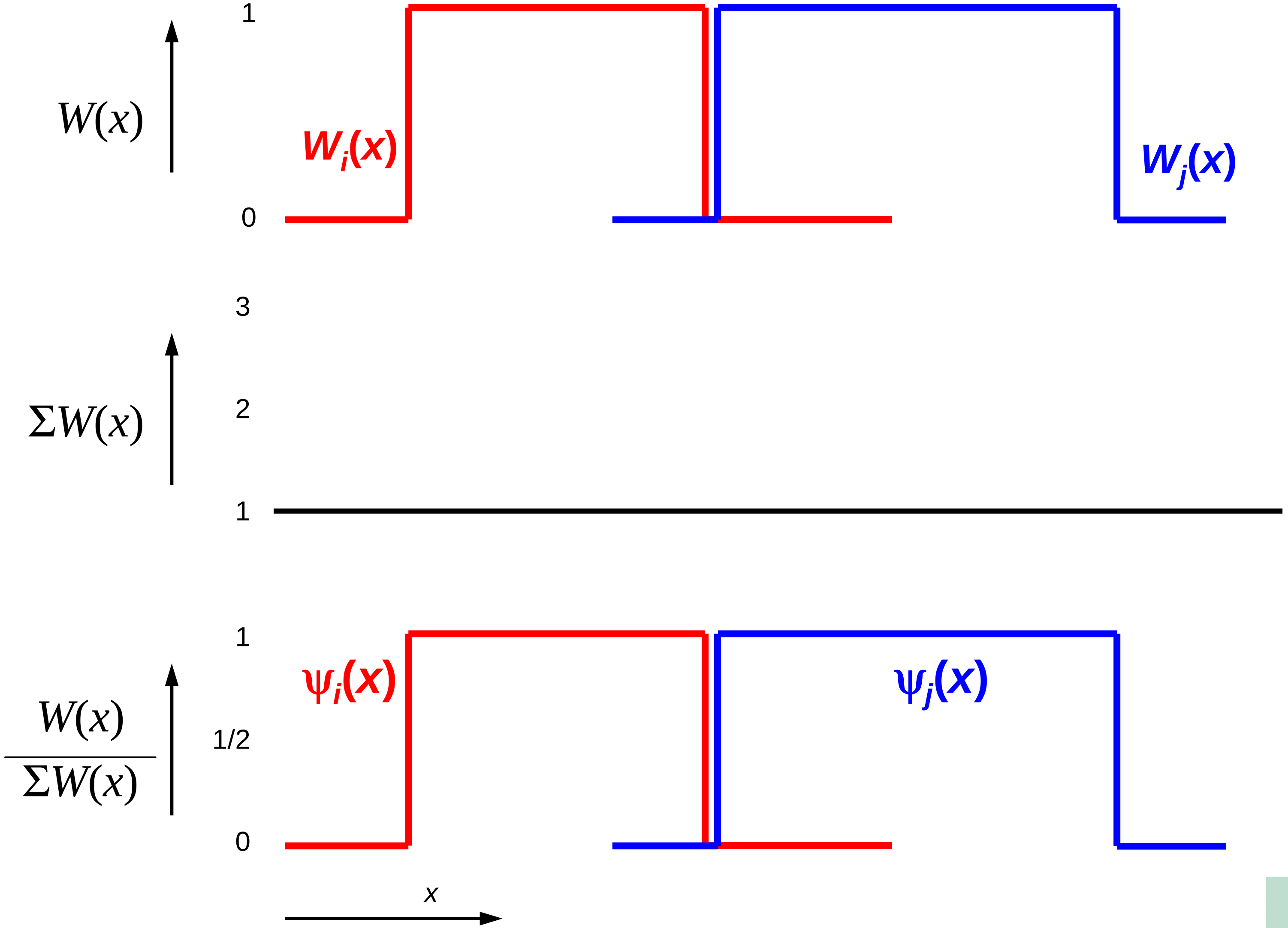
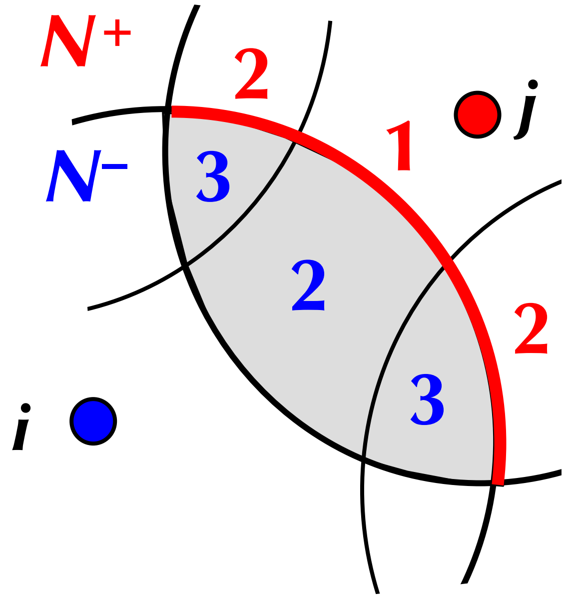
Top-hat kernel functions



Non-overlapping top-hats = mesh finite volume

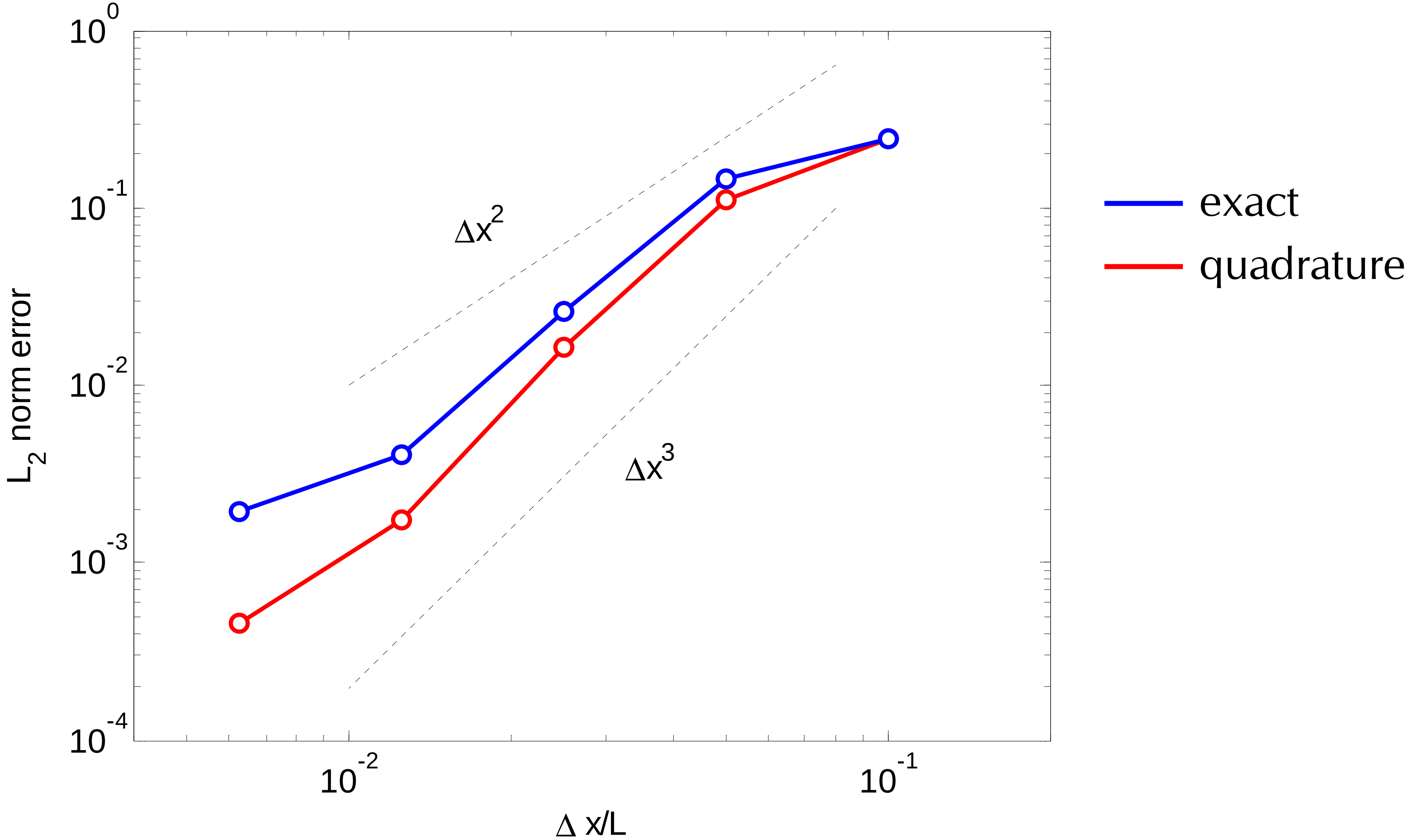


Evaluation of β_{ij} with overlap top-hat kernel



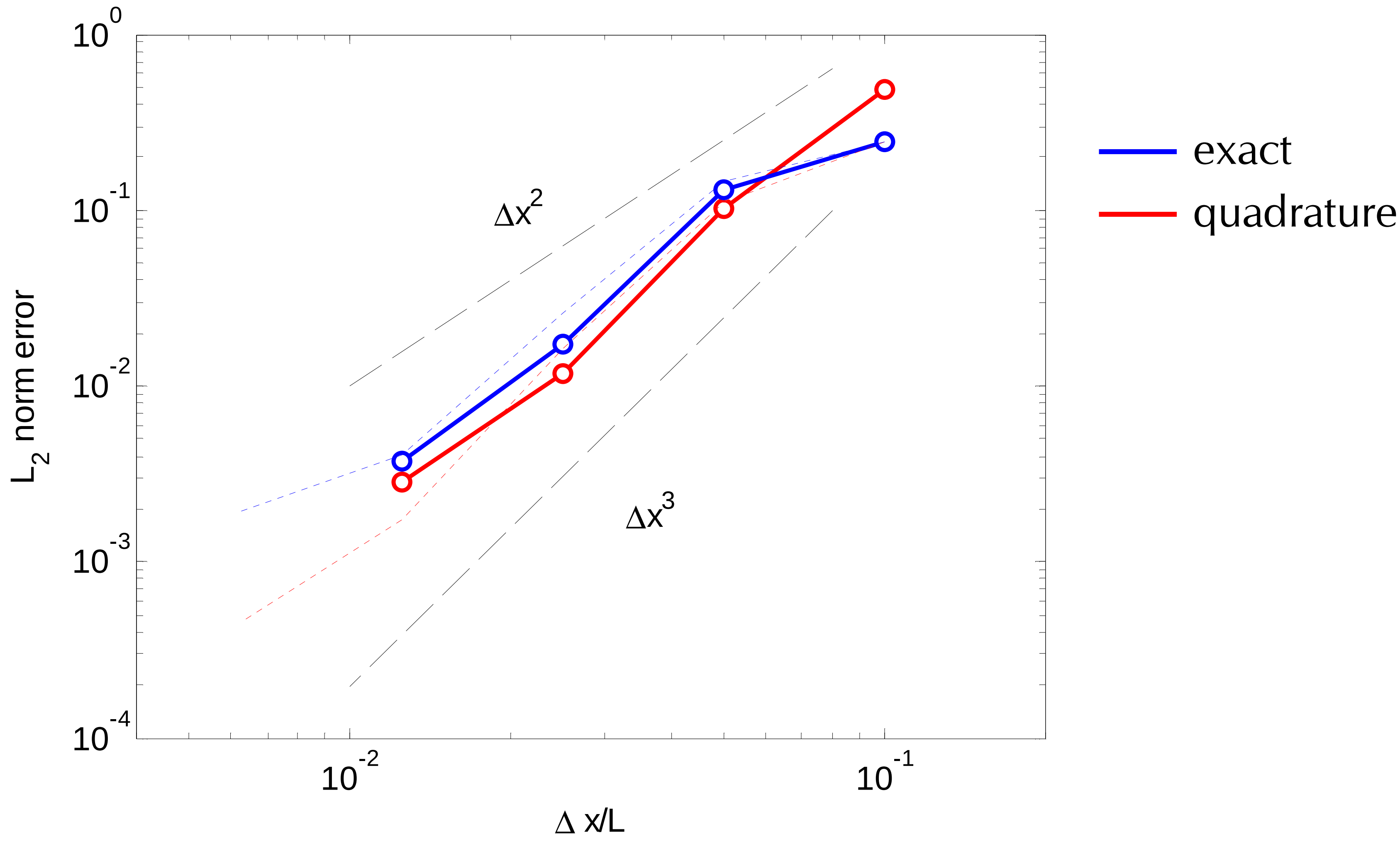
$$\int \frac{W_j \nabla W_i}{\left(\sum_k W_k(\mathbf{x})\right)^2} \mathrm{d}\mathbf{x} = \int \frac{W_j \nabla W_i}{N(\mathbf{x})^2} \mathrm{d}\mathbf{x} \quad = \quad \int \left(\frac{1}{N^-(x)} - \frac{1}{N^+(x)} \right) \mathbf{n} \, \mathrm{d}s$$

Comparison of integration methods



Taylor-Green flow
Re = 100, Eulerian particles

Comparison of integration methods



Taylor-Green flow
Re = 100, nearly Lagrangian particles

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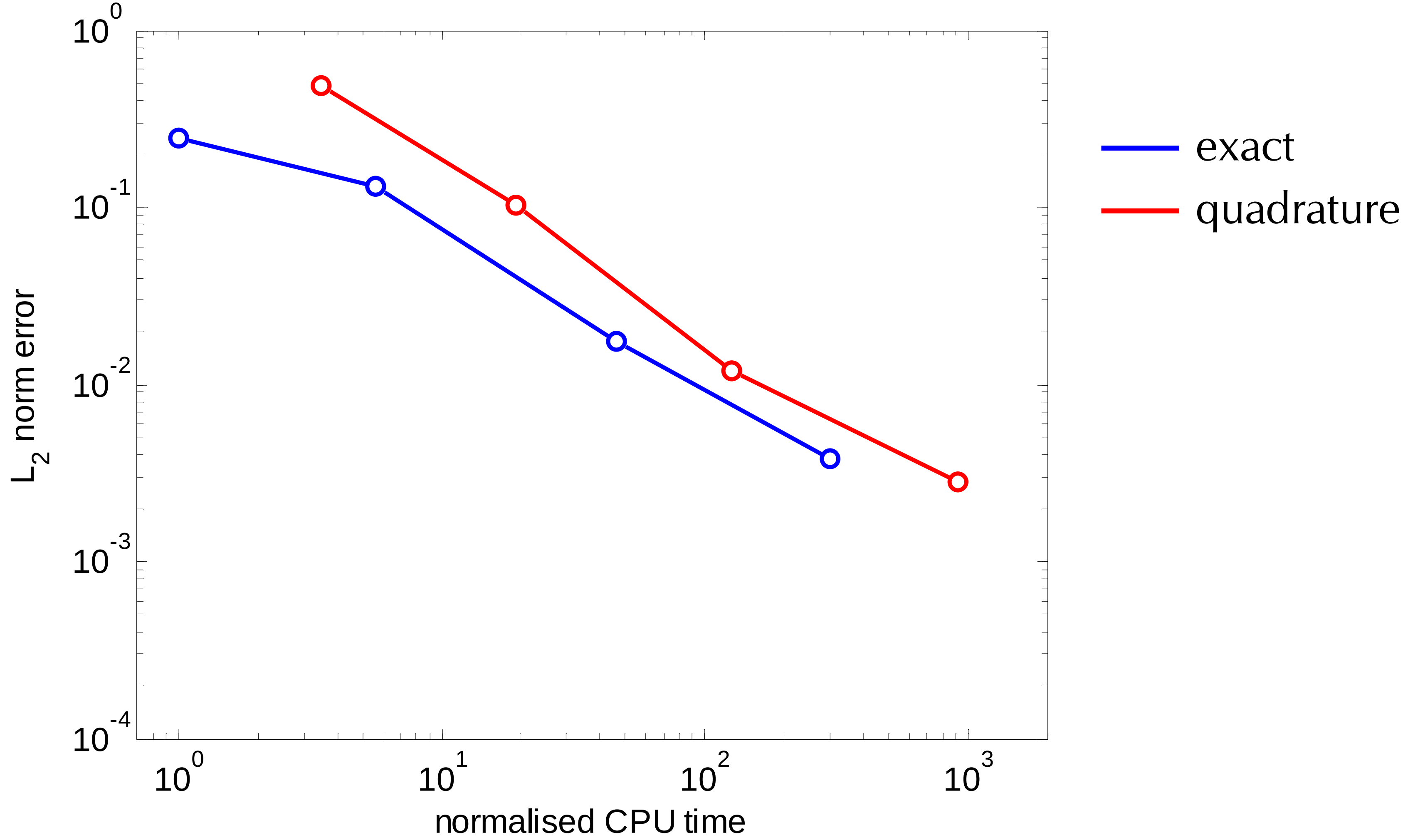
Comparison of integration methods



Taylor-Green flow

Re = 100, nearly Lagrangian particles

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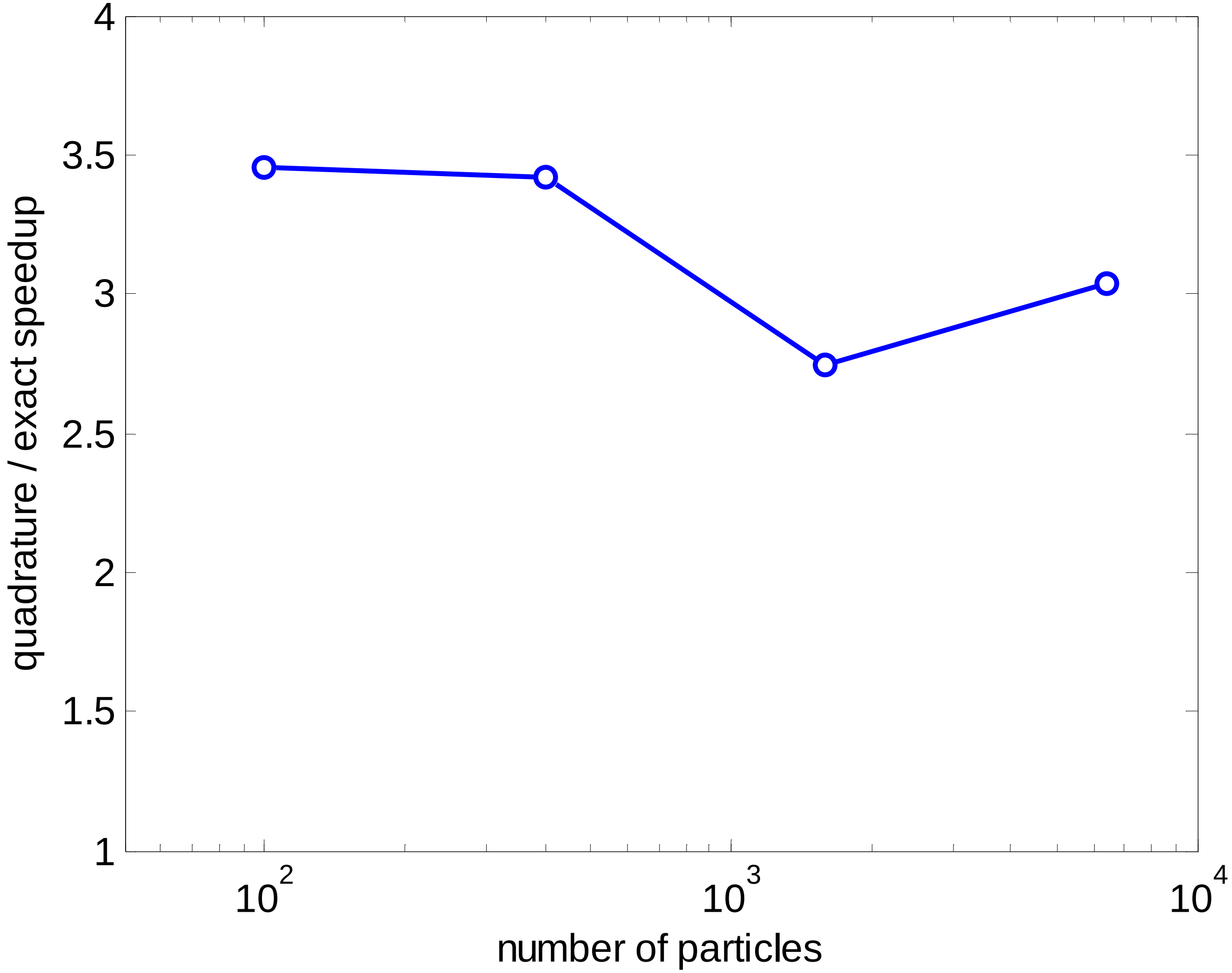
Comparison of integration methods



Taylor-Green flow

Re = 100, nearly Lagrangian particles

Kernels for FVPM: summary

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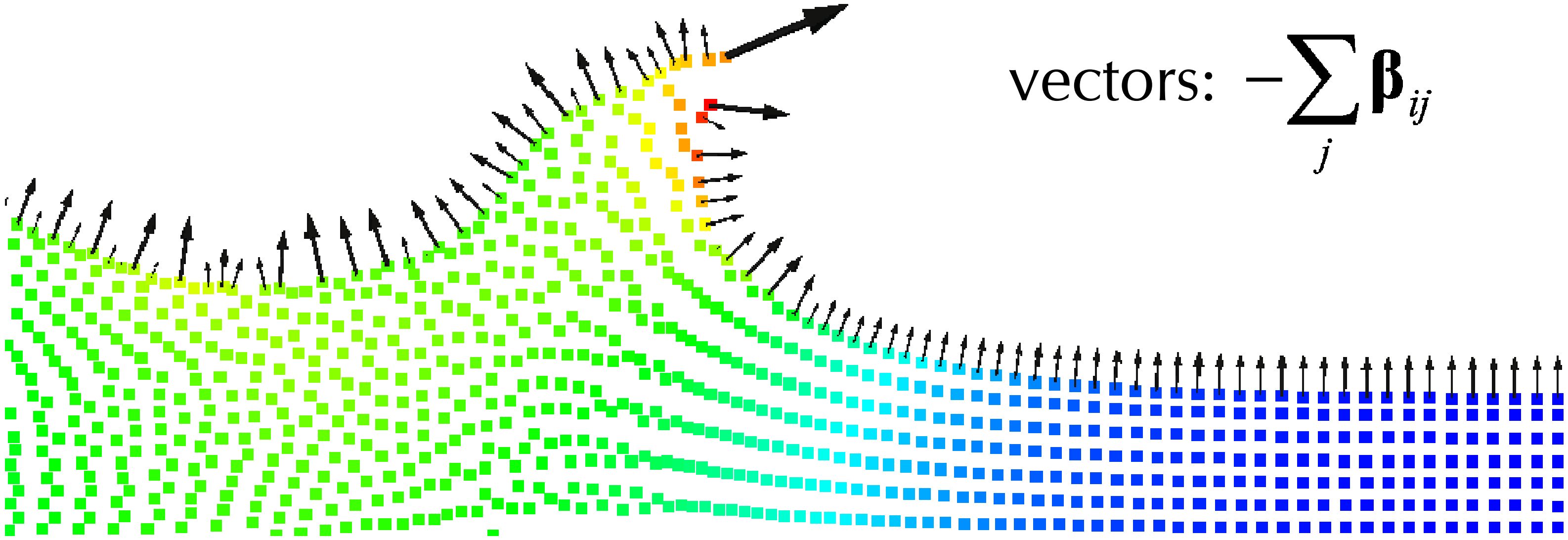
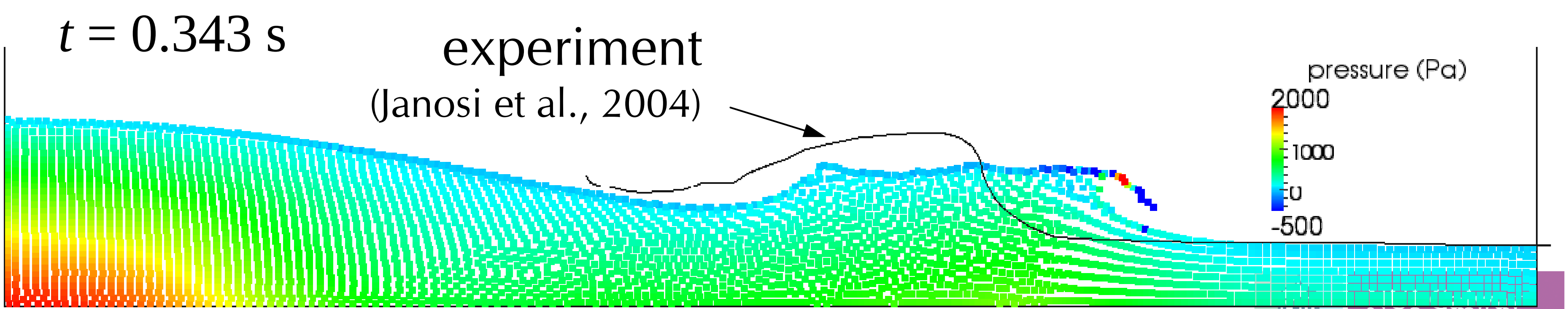
Exact β_{ij} enables free-surface modelling

SPHERIC benchmark 5



vectors: $-\sum_j \boldsymbol{\beta}_{ij}$



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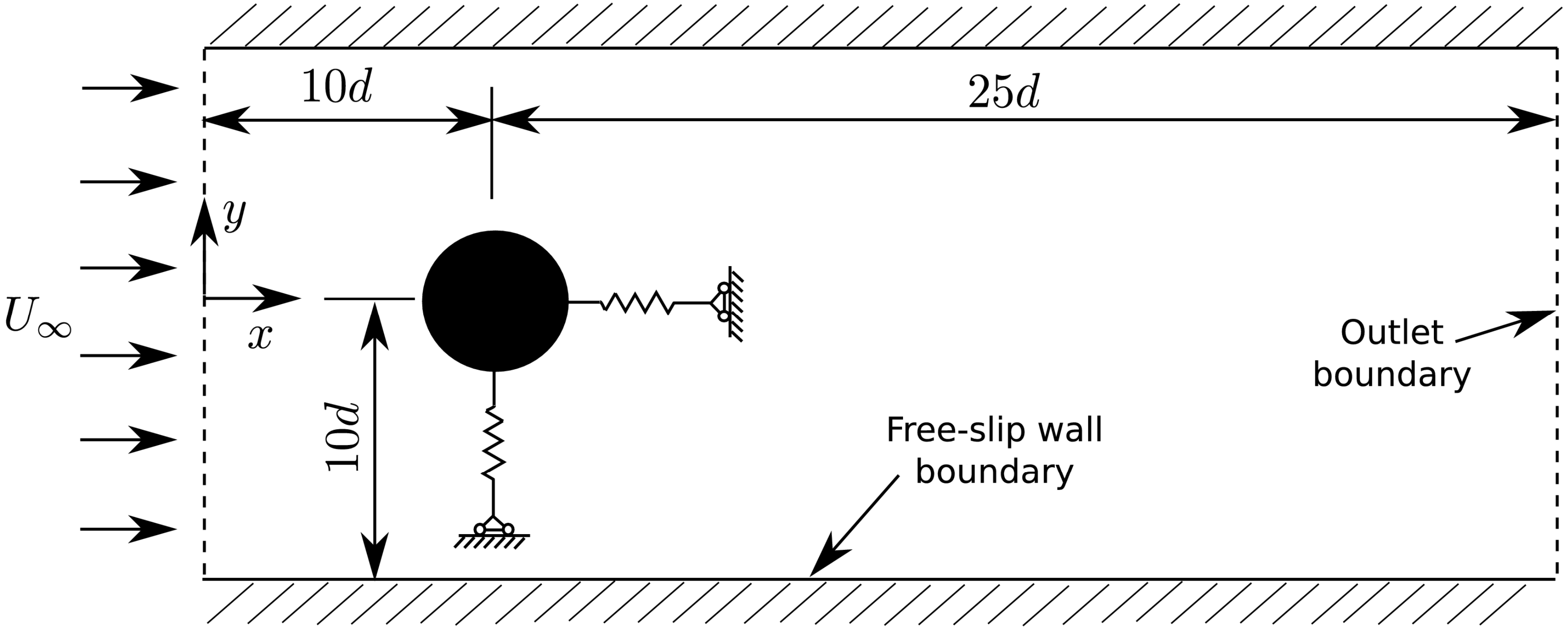
Vortex-induced vibration



$\mathrm{Re}_d = 100$

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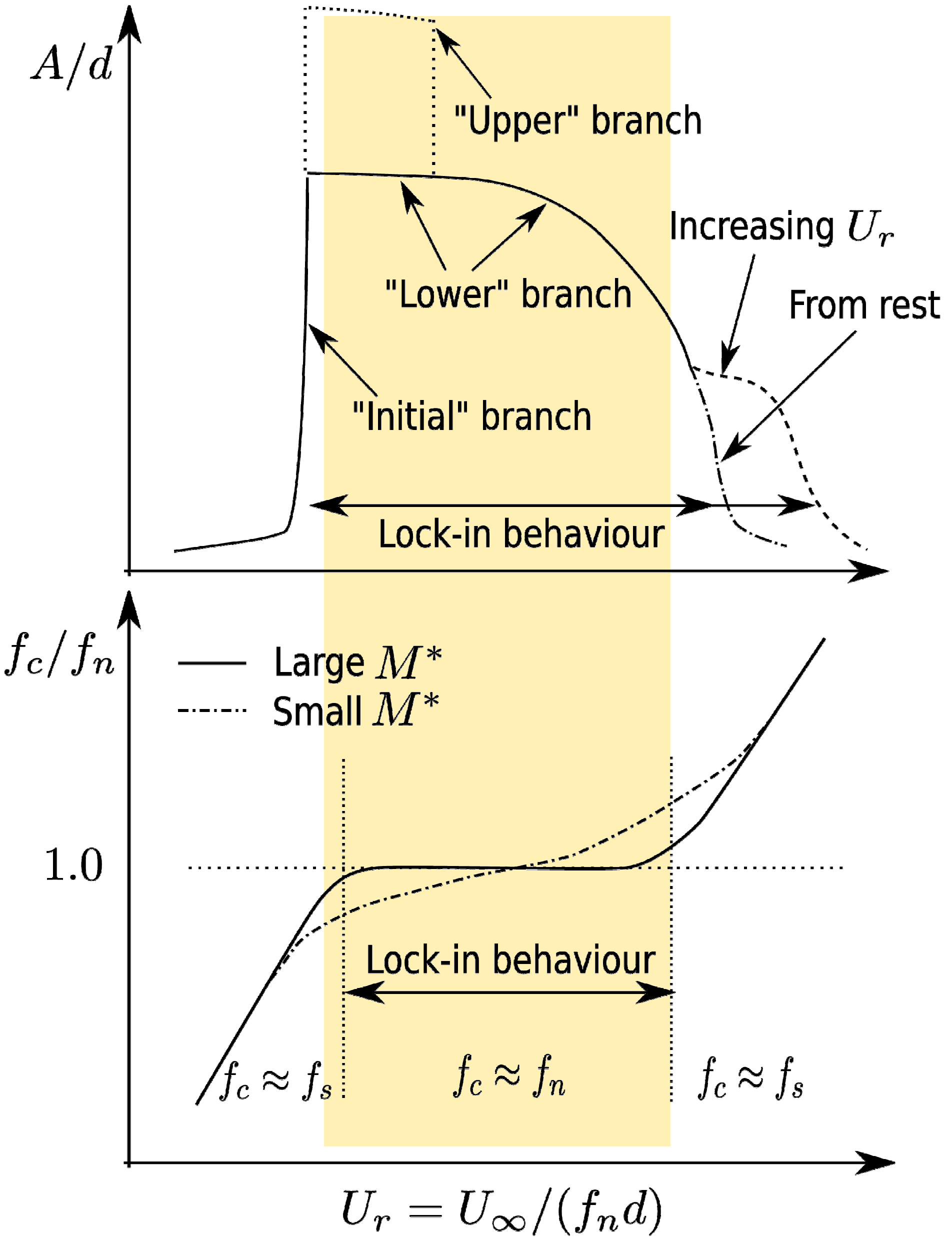
Vortex-induced vibration



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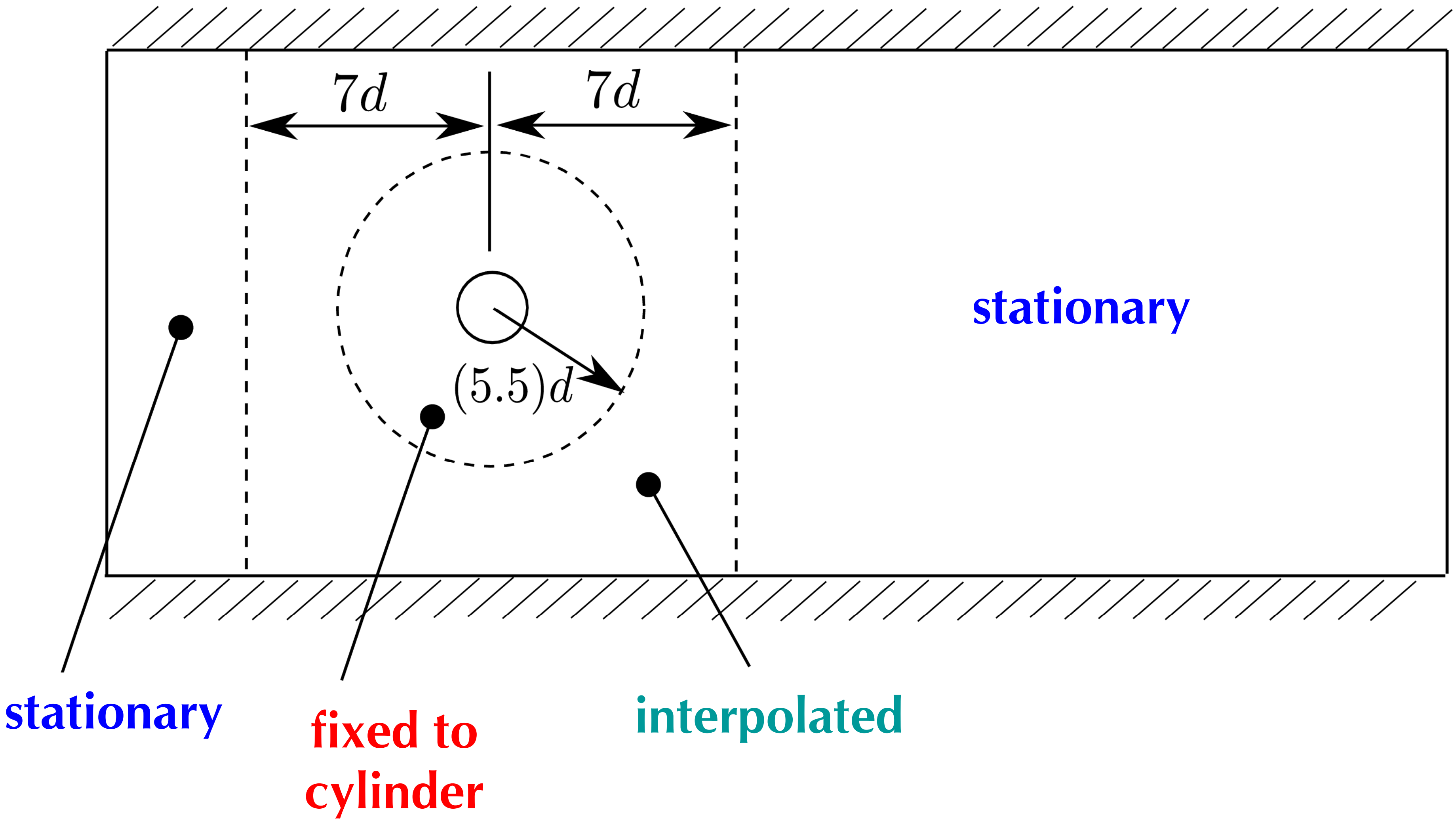
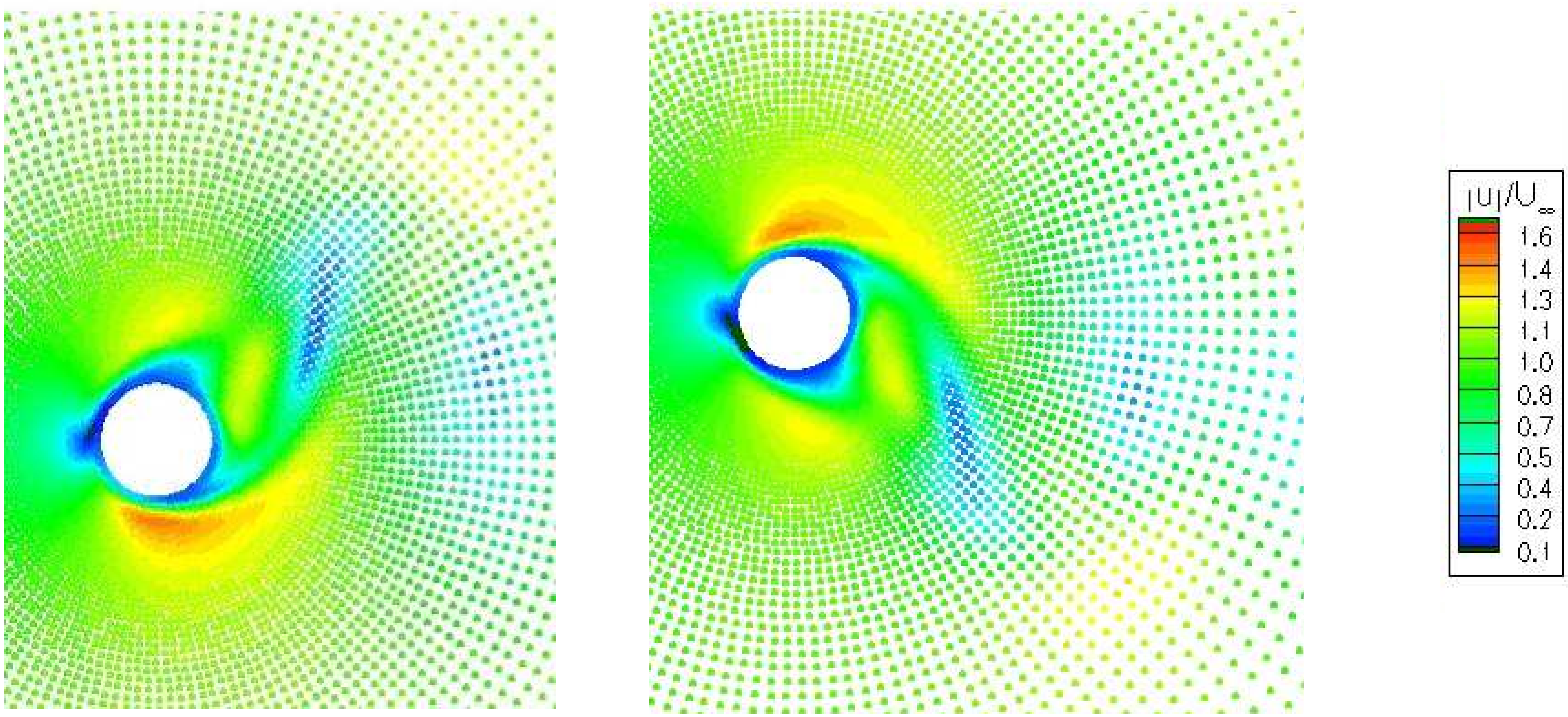
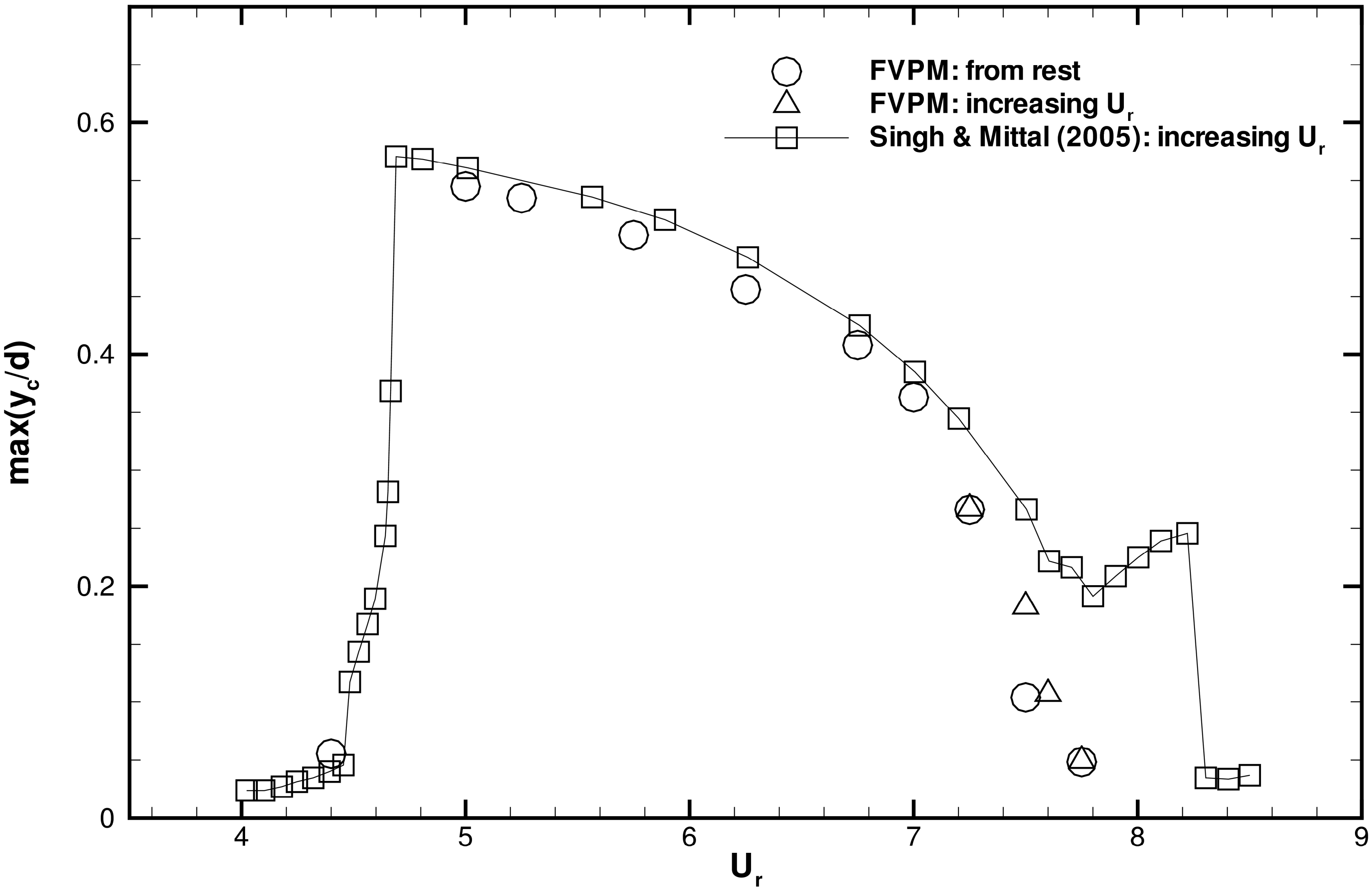
Particle motion schemes



Results



Results



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Conclusions

- FVPM is closely linked to Riemann SPH
- FVPM gives robust, simple boundary treatments
- Exact interaction vectors yield $3\times$ speedup
- Validated for bodies with prescribed and free motion

Future work

- Control of particle motion and distribution is critical
- Extension to 3D

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SEVENTH FRAMEWORK PROGRAMME

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IRCSET Irish Research Council for Science, Engineering & Technology
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