Smoothed Particle Hydrodynamics

Or how I learnt to stop worrying and love Lagrangians

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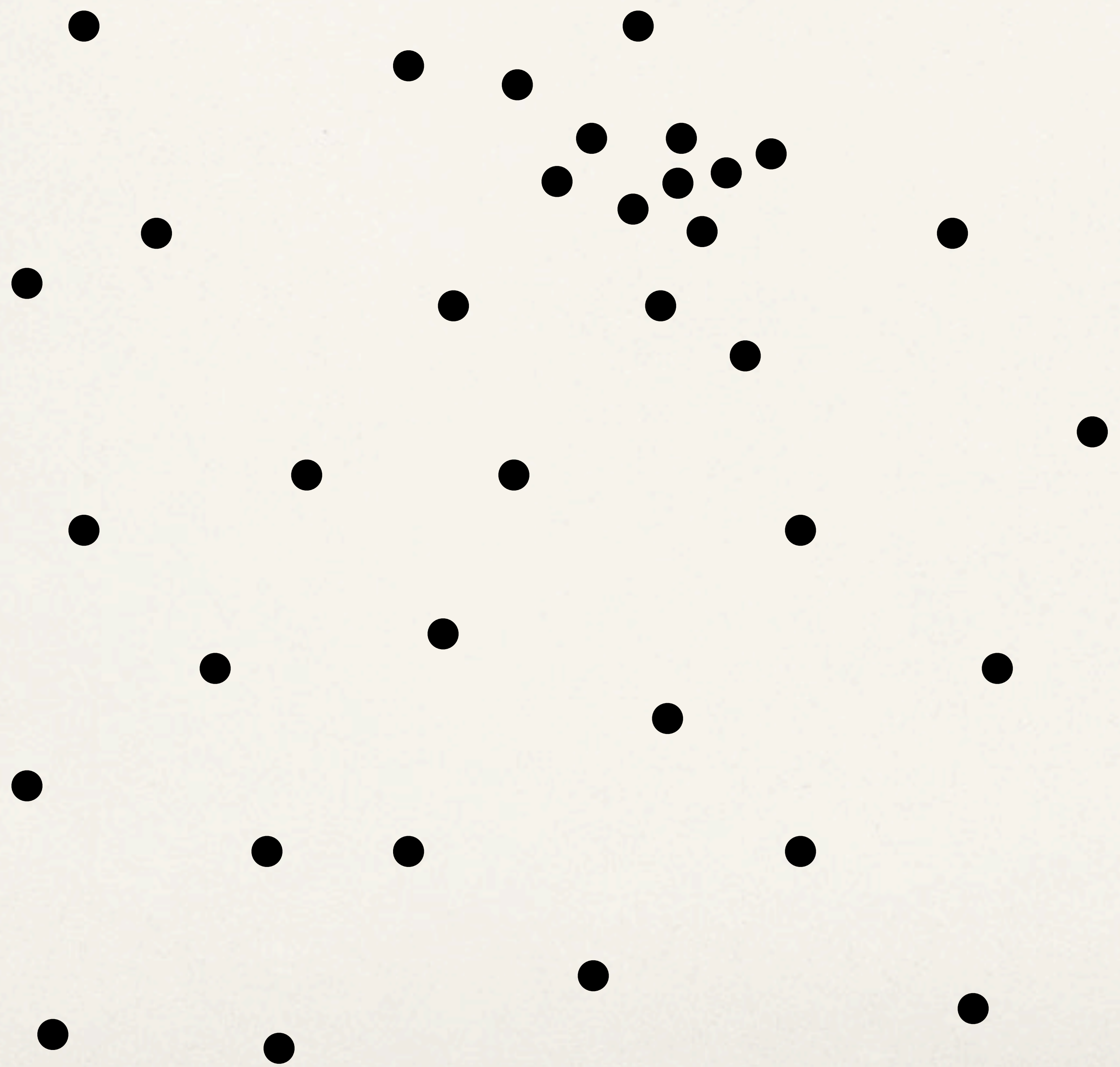
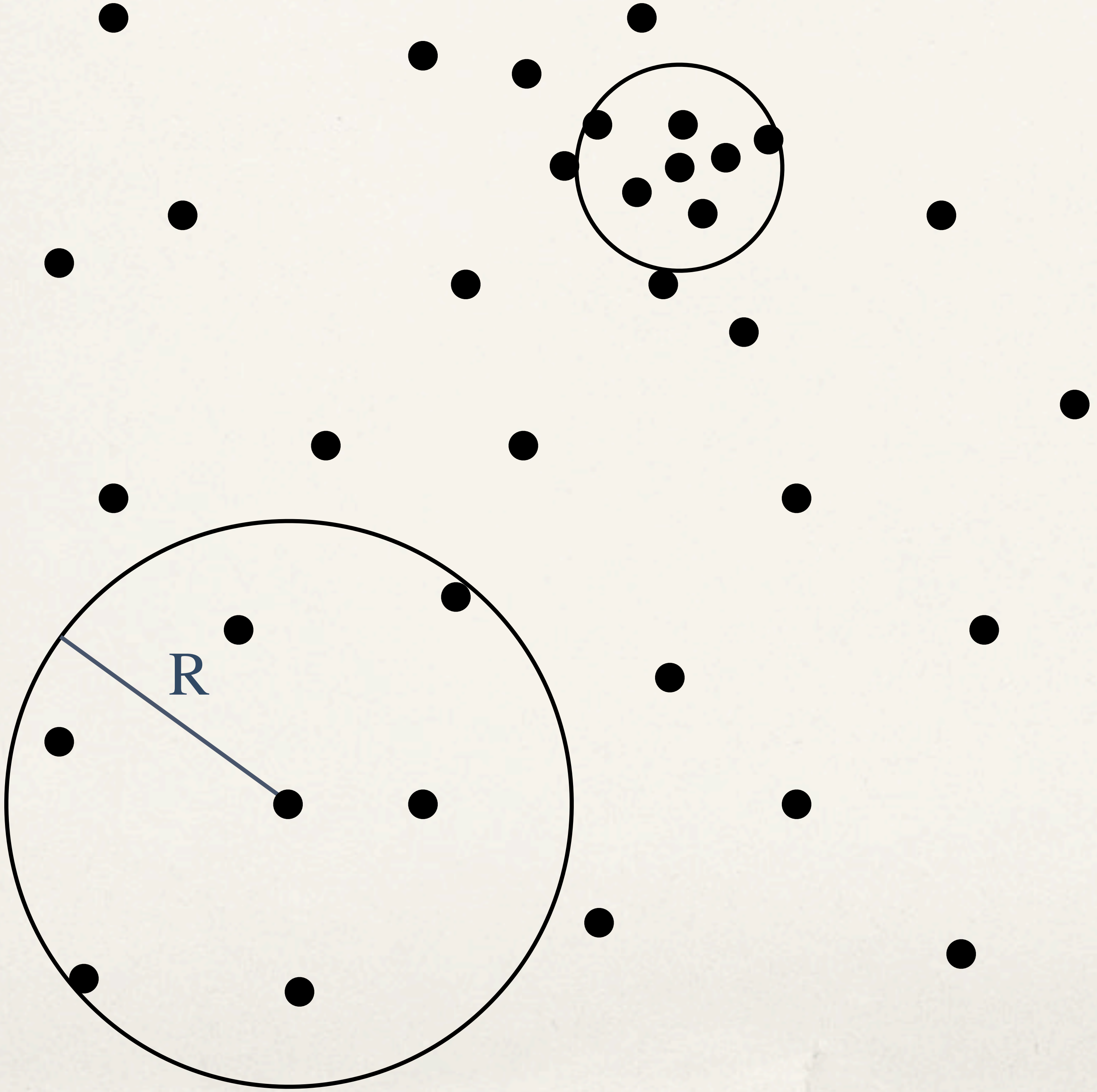
Keynote @ SPHERIC VIII, June 4th-6th 2013, Trondheim, Norway

SPH starts here…



What is the density?

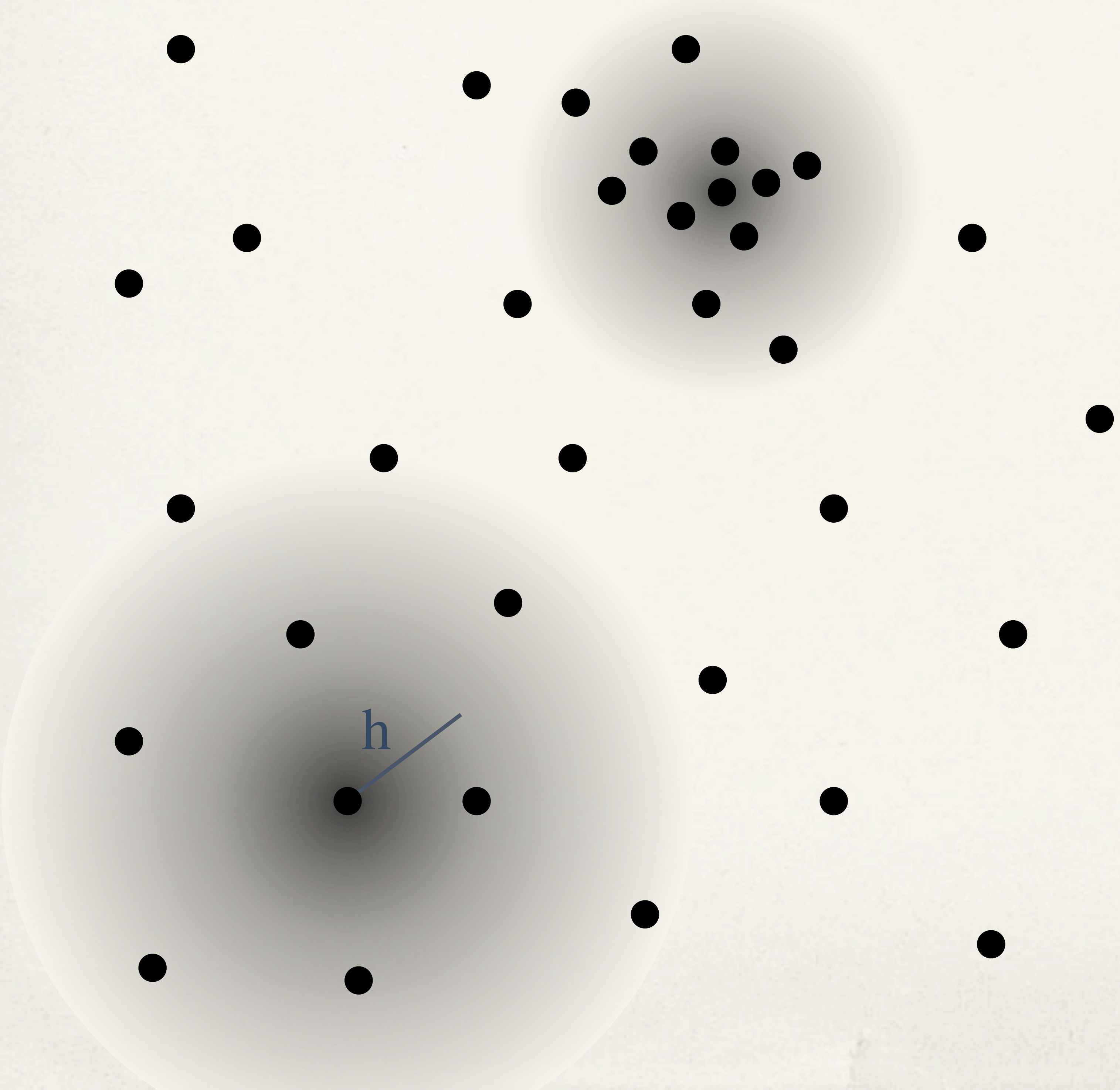
Not the SPH density estimate



$$\rho_a = \frac{\sum_b m_b}{V_a}$$

NOT SPH

The SPH density estimate



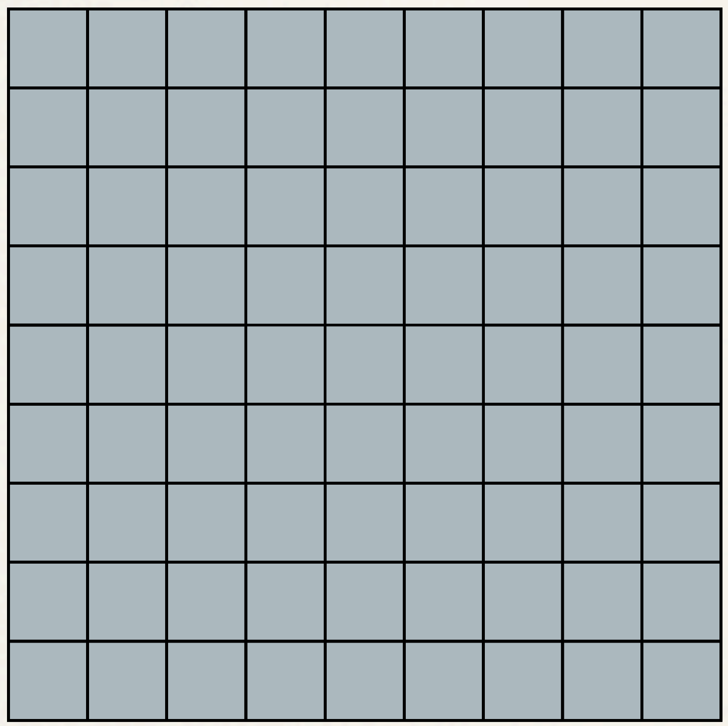
Kernel-weighted sum:

$$\rho(\mathbf{r}) = \sum_{j=1}^{N} m_j W(|\mathbf{r} - \mathbf{r}_j|, h)$$

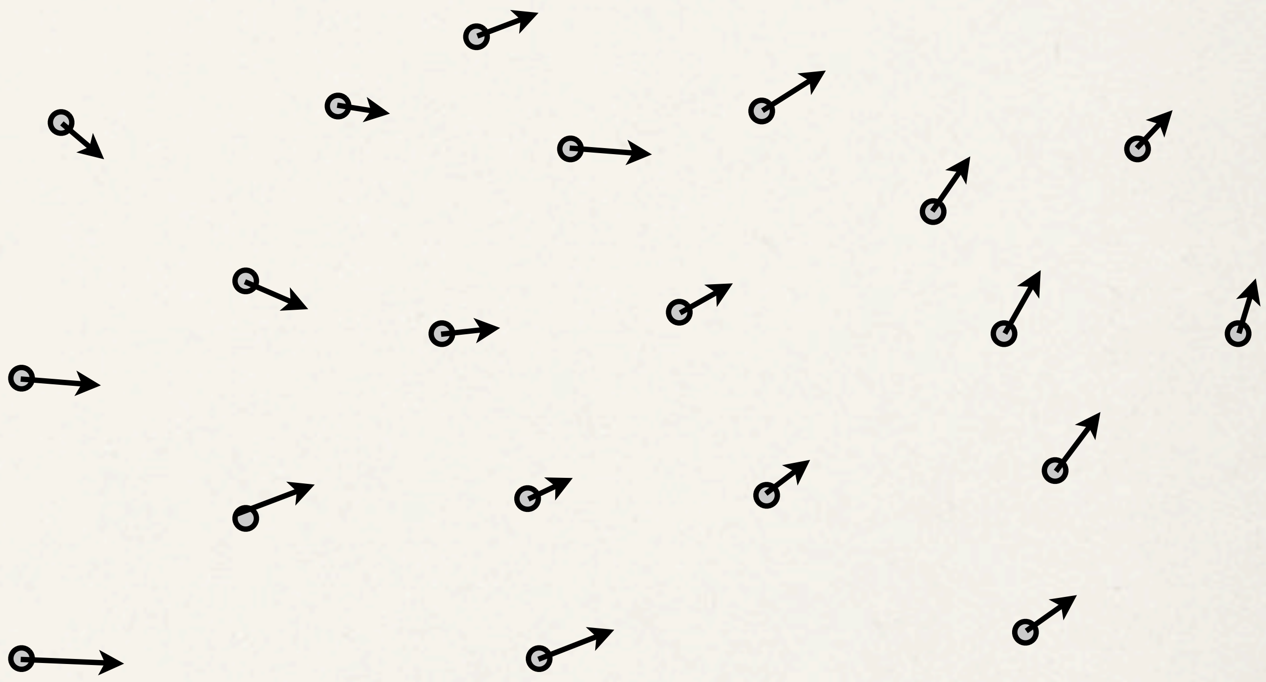
e.g. $W = \frac{\sigma}{h^3} e^{-r^2/h^2}$

Resolution follows mass

Grid



SPH



$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}$$

From density to hydrodynamics

$$L_{sph} = \sum_j m_j \left[\frac{1}{2} v_j^2 - u_j(\rho_j, s_j) \right]$$ ← Lagrangian

$+$

$$\mathrm{d}u = \frac{P}{\rho^2} \mathrm{d}\rho$$ ← 1st law of thermodynamics

$+$

$$\nabla \rho_i = \sum_j m_j \nabla W_{ij}(h)$$ ← density sum

$+$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \mathbf{v}} \right) - \frac{\partial L}{\partial \mathbf{r}} = 0$$ ← Euler-Lagrange equations

$=$

$$\frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = - \sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla_i W_{ij}(h)$$ ← equations of motion!

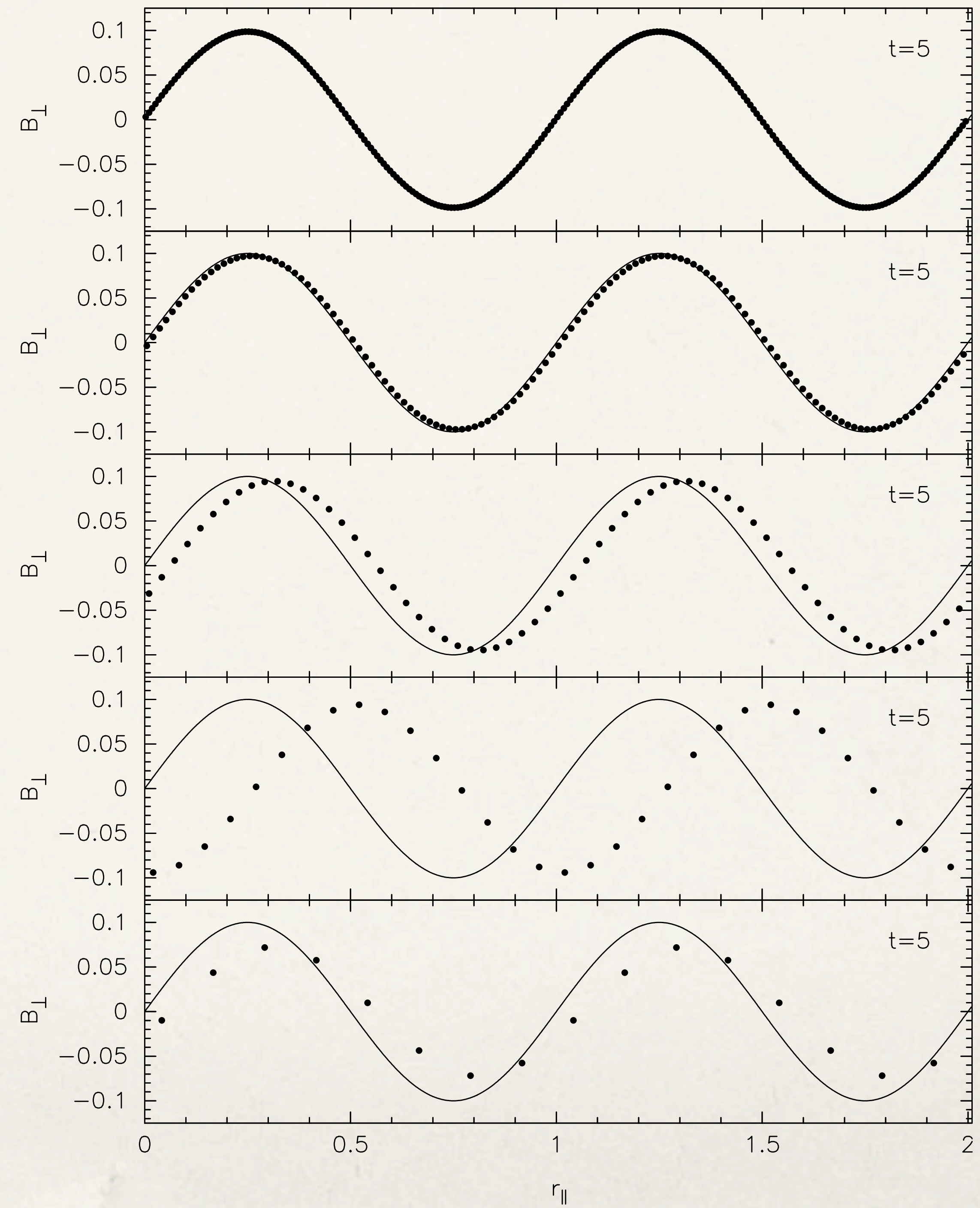
$$\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = - \frac{\nabla P}{\rho} \right)$$

What this gives us: Advantages of SPH

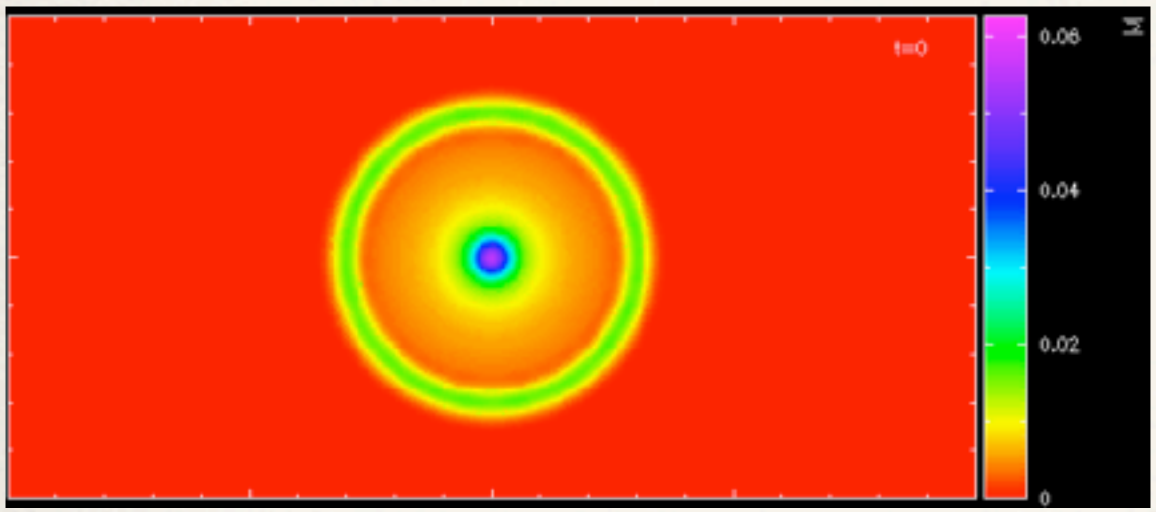
- An exact solution to the continuity equation
- Resolution follows mass
- ZERO dissipation
- Advection done perfectly
- EXACT conservation of mass, momentum, angular momentum, energy and entropy
- A guaranteed minimum energy state

Zero dissipation - Example I.

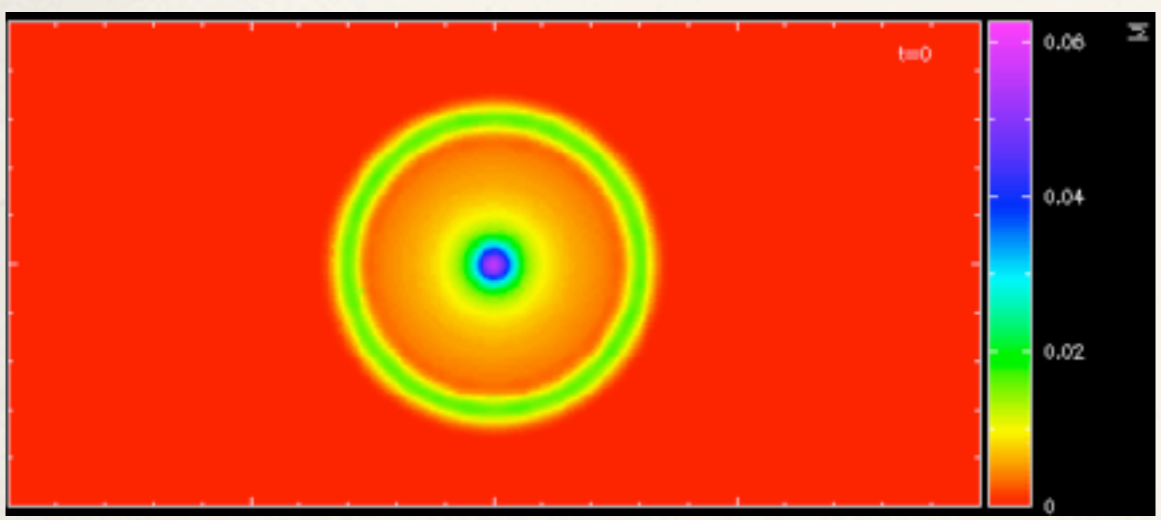
Propagation of a circularly polarised Alfvén wave



Zero dissipation - example II: Advection of a current loop



first 25 crossings



1000 crossings (Rosswog & Price 2007)

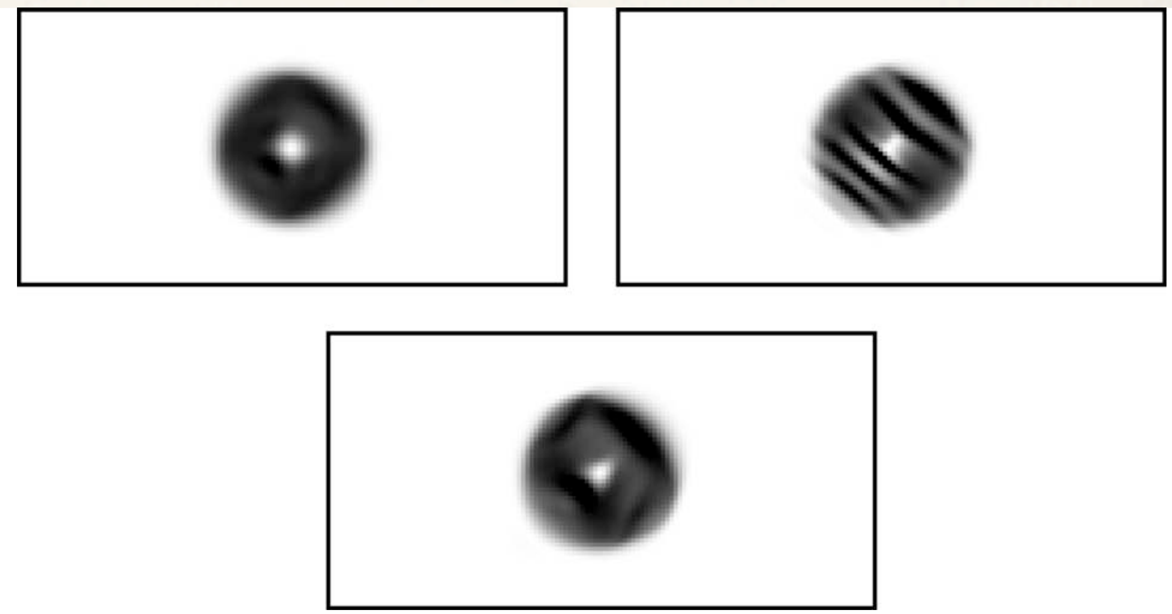


Fig. 3. Gray-scale images of the magnetic pressure $(B_x^2 + B_y^2)$ at $t = 2$ for an advected field loop ($v_0 = \sqrt{5}$) using the $\mathcal{E}_z^{\alpha}$ (top left), (top right) and $\mathcal{E}_z^{c}$ (bottom) CT algorithm.

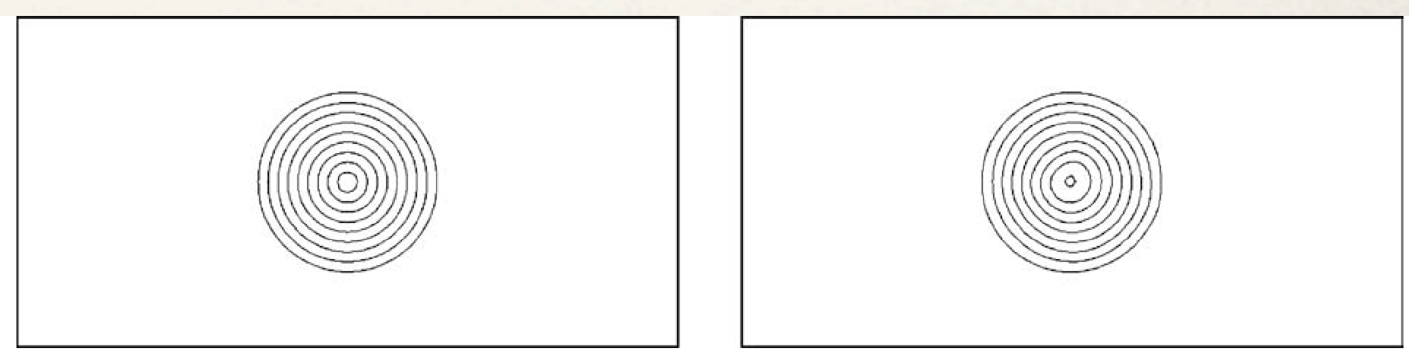


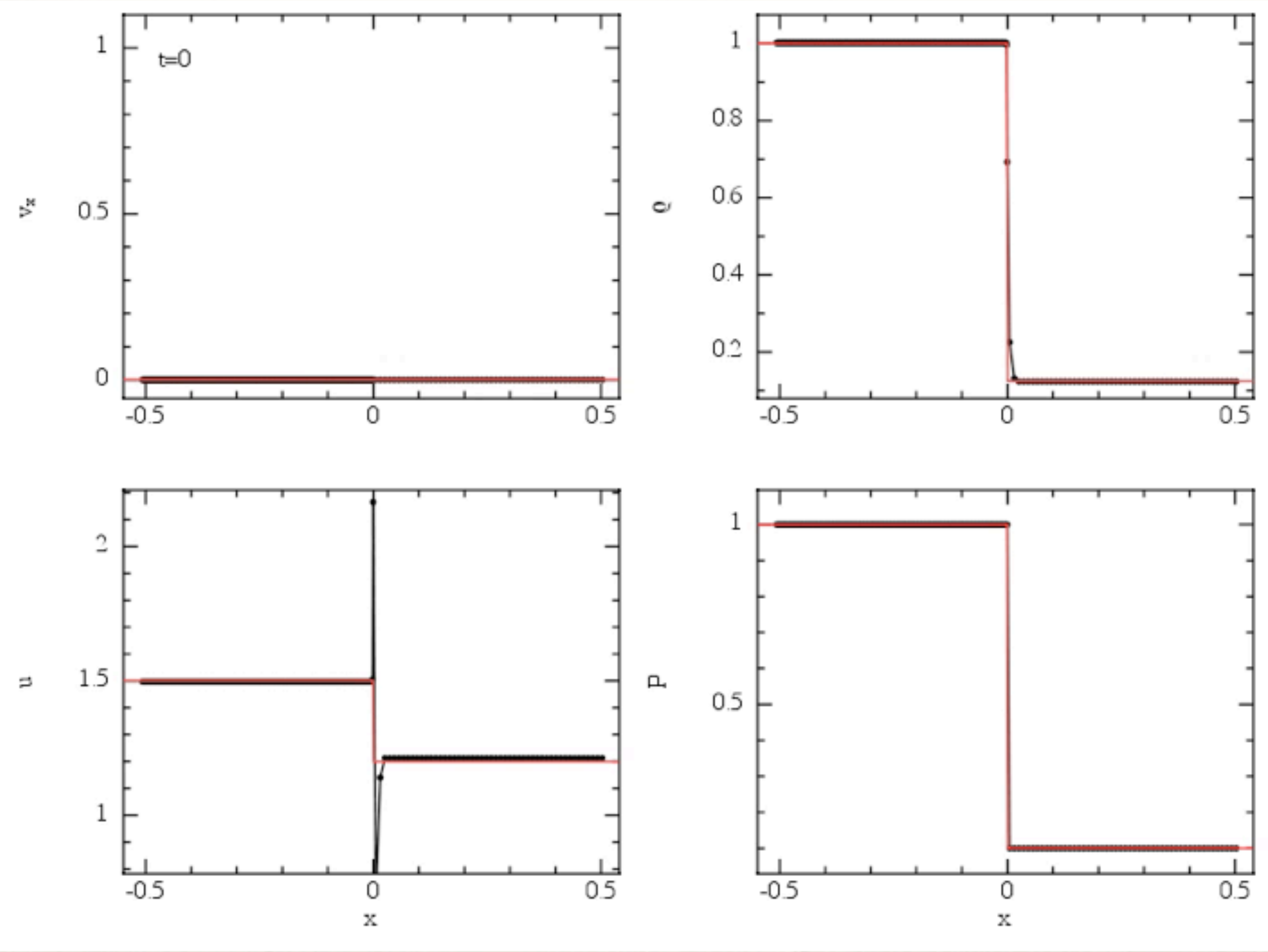
Fig. 8. Magnetic field lines at $t = 0$ (left) and $t = 2$ (right) using the CTU + CT integration algorithm.

2 crossings (Gardiner & Stone 2005)

SPH

grid

Zero dissipation...



From density to hydrodynamics

$$L_{sph} = \sum_j m_j \left[\frac{1}{2} v_j^2 - u_j(\rho_j, s_j)\right]$$ ← Lagrangian

$+$

$$\mathrm{d}u = \frac{P}{\rho^2}\mathrm{d}\rho$$ ← 1st law of thermodynamics

Here we assume that density is differentiable and that the entropy does not change

$+$

$$\nabla\rho_i = \sum_j m_j \nabla W_{ij}(h)$$ ← density sum

$+$

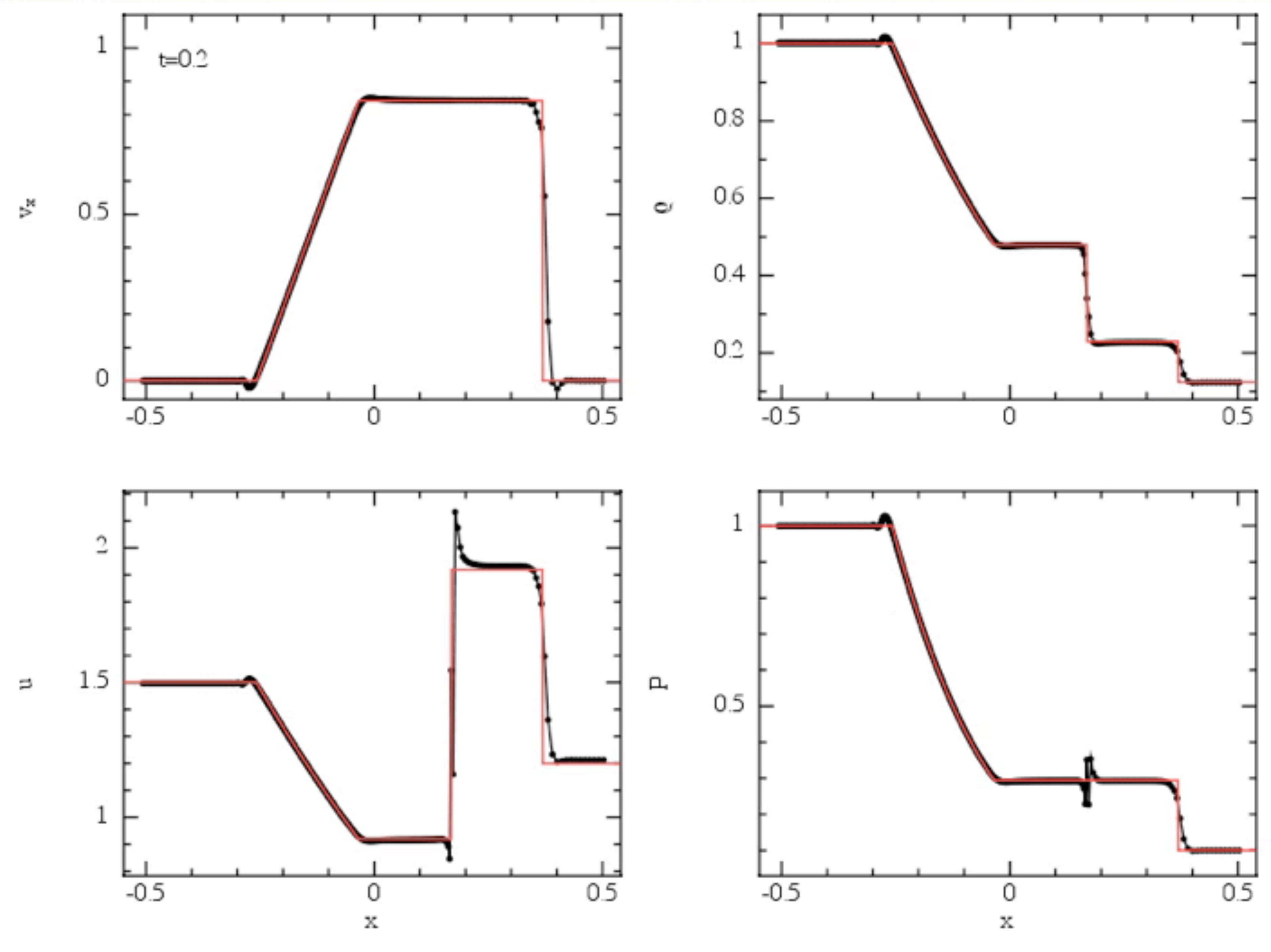
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{r}} = 0$$ ← Euler-Lagrange equations

$=$

$$\frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2}\right)\nabla_i W_{ij}(h)$$ ← equations of motion!

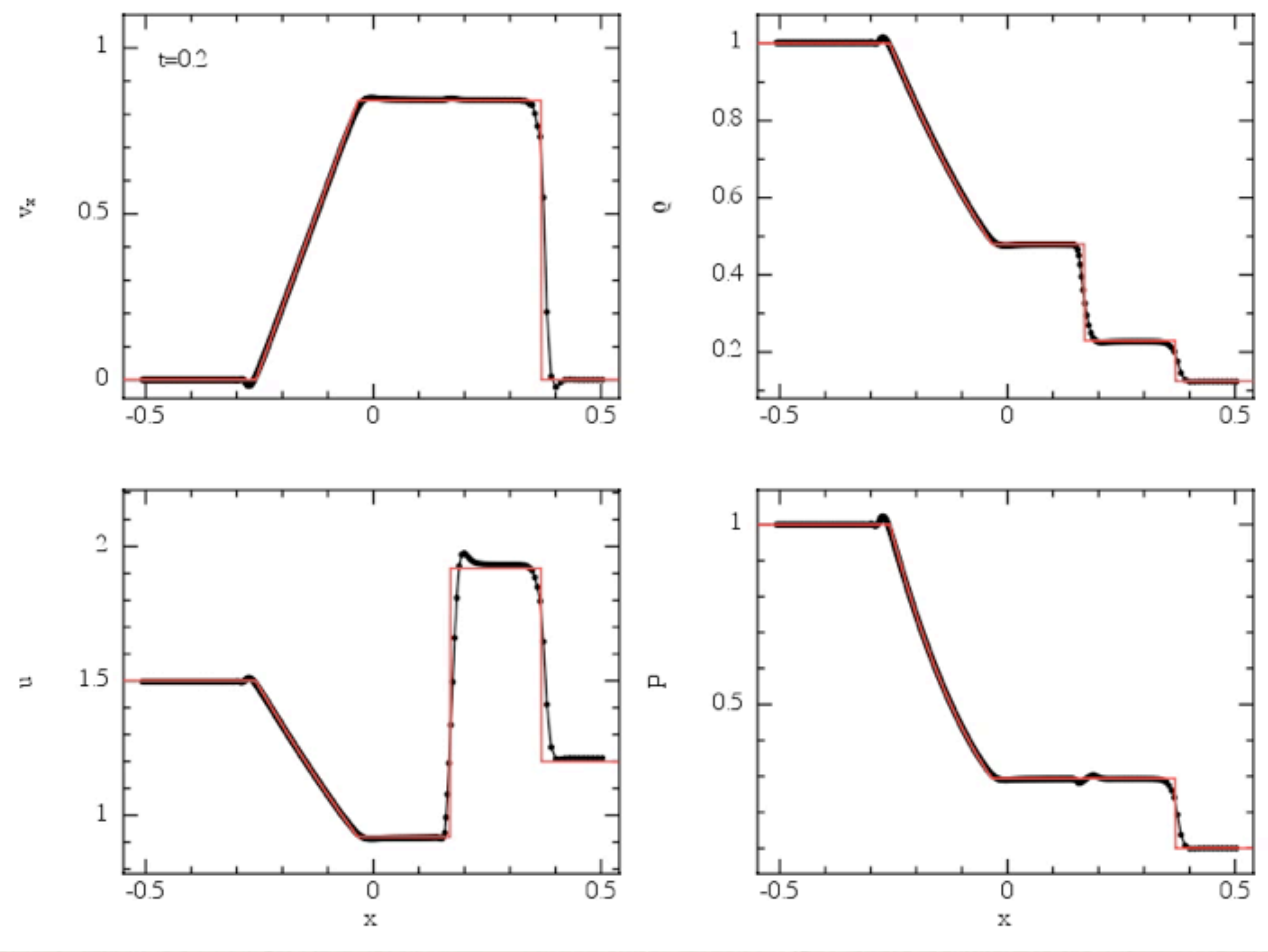
$$\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\nabla P}{\rho}\right)$$

Zero dissipation...



...so we have to add some

Must treat EVERY discontinuity



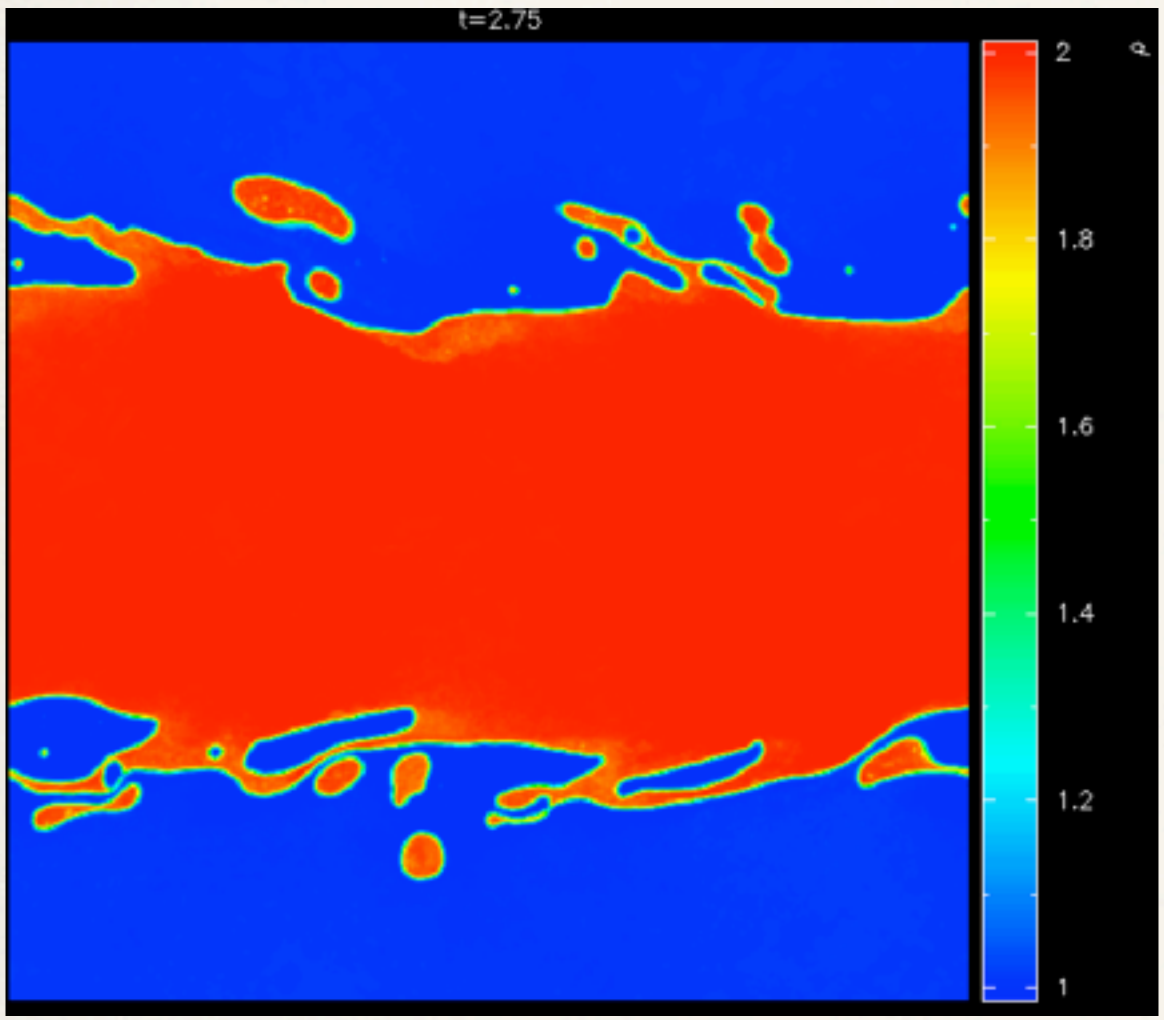
Viscosity

+

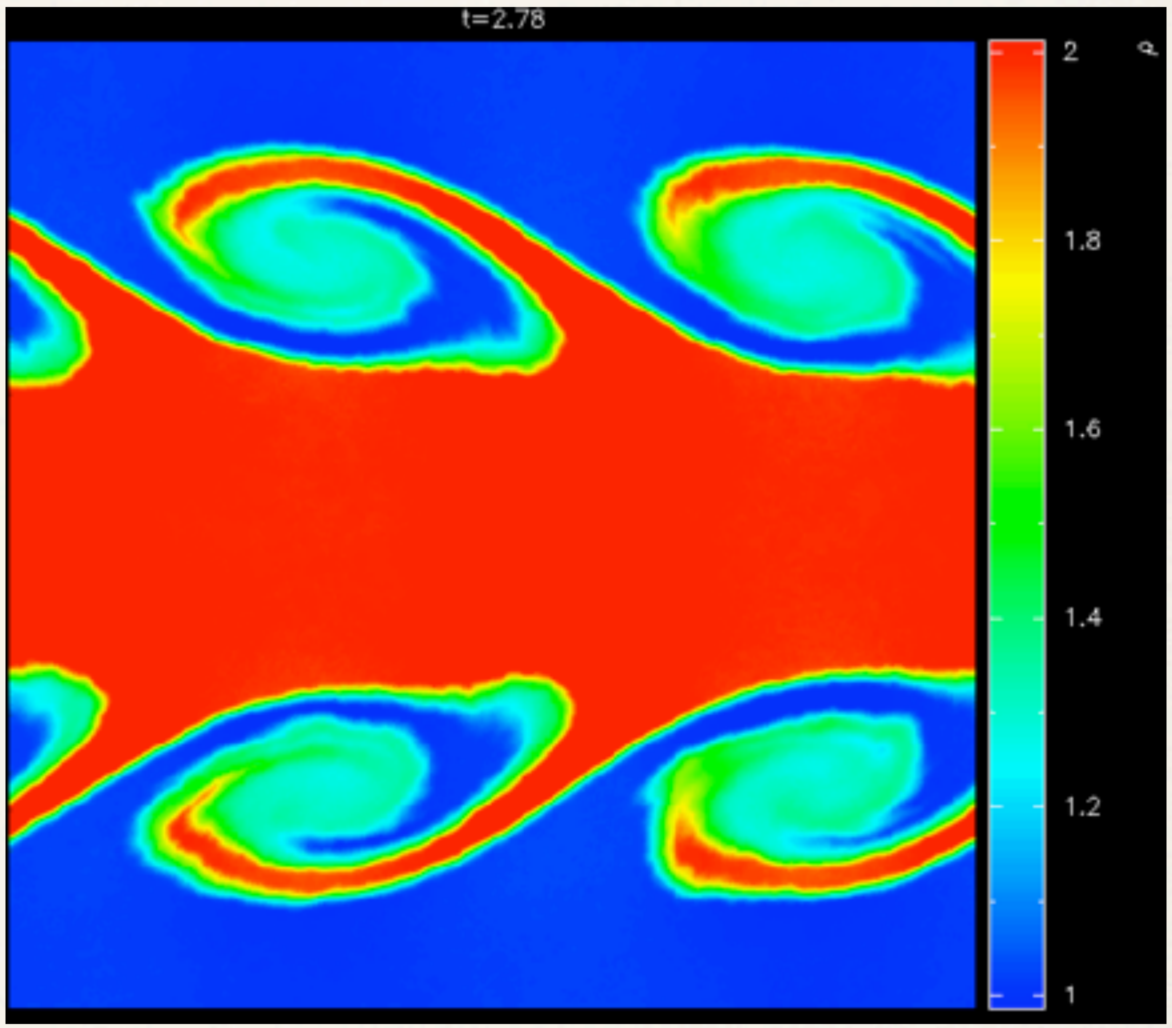
Conductivity

Must treat discontinuities properly...

Agertz et al. 2007, Price 2008 and too many others



Viscosity only



Viscosity + conductivity

This issue has nothing to do with the instability itself
It is related to the treatment of the contact discontinuity

dissipation terms need to be explicitly added


OFF

The key is a good switch

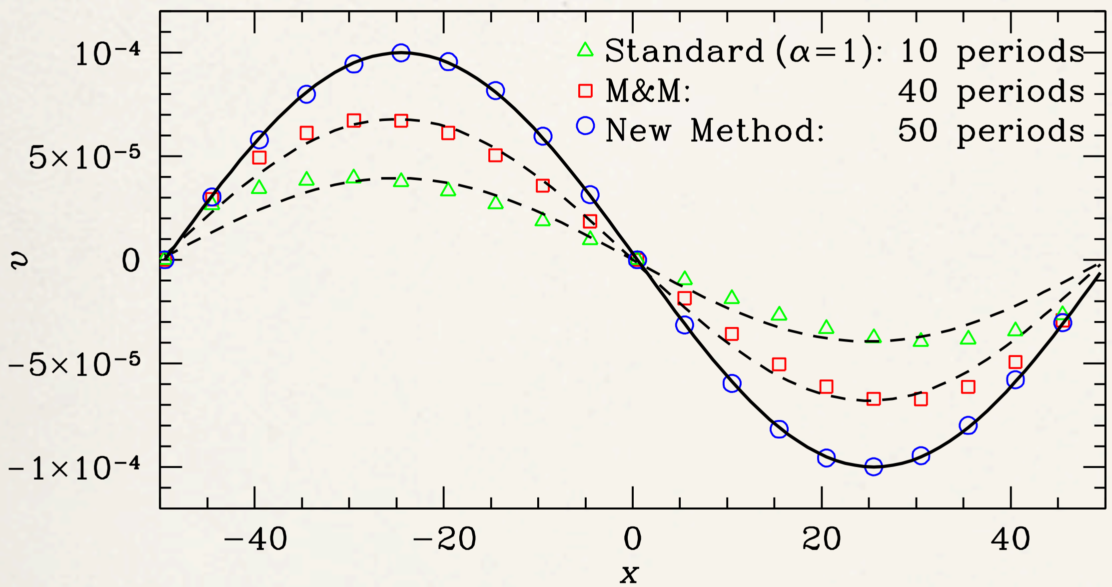


Figure 2. As Fig. 1, but for SPH with standard ($\alpha = 1$) or Morris & Monaghan (1997) artificial viscosity, as well as our new method (only every fifth particle is plotted). Also shown are the undamped wave (*solid*) and lower-amplitude sinusoidals (*dashed*). Only with our method the wave propagates undamped, very much like SPH without any viscosity, as in Fig. 1.

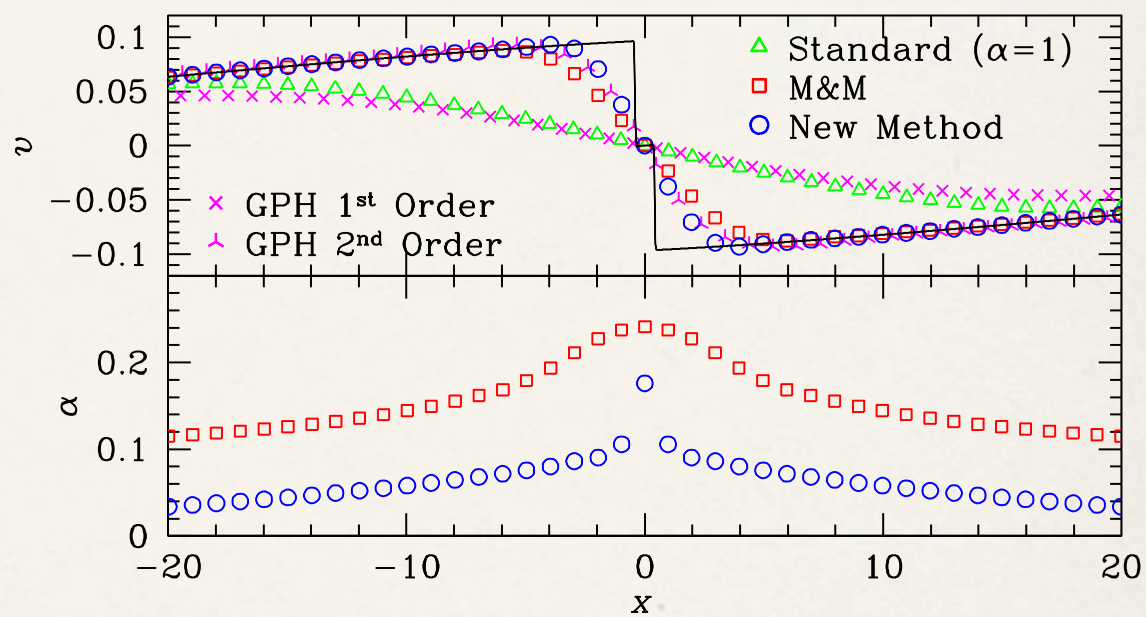


Figure 6. Steepening of a 1D sound wave: velocity and viscosity parameter vs. position for standard SPH, the M&M method, our new scheme, and Godunov particle hydrodynamics of first and second order (GPH, Cha & Whitworth 2003), each using 100 particles per wavelength. The solid curve in the top panel is the solution obtained with a high-resolution grid code.

Switch for artificial viscosity: Cullen & Dehnen (2010)

What this gives us: Advantages of SPH

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- Resolution follows mass
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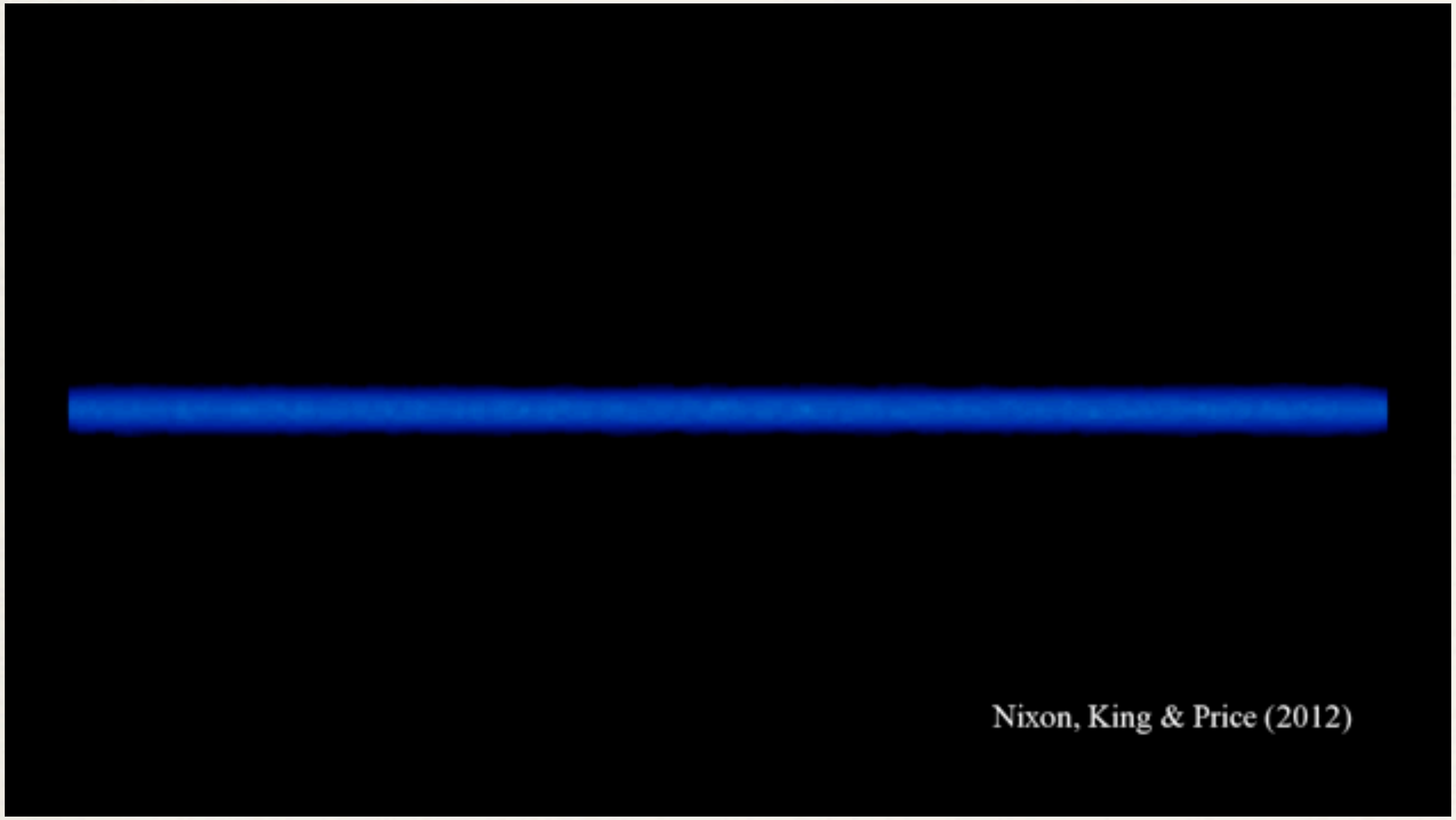
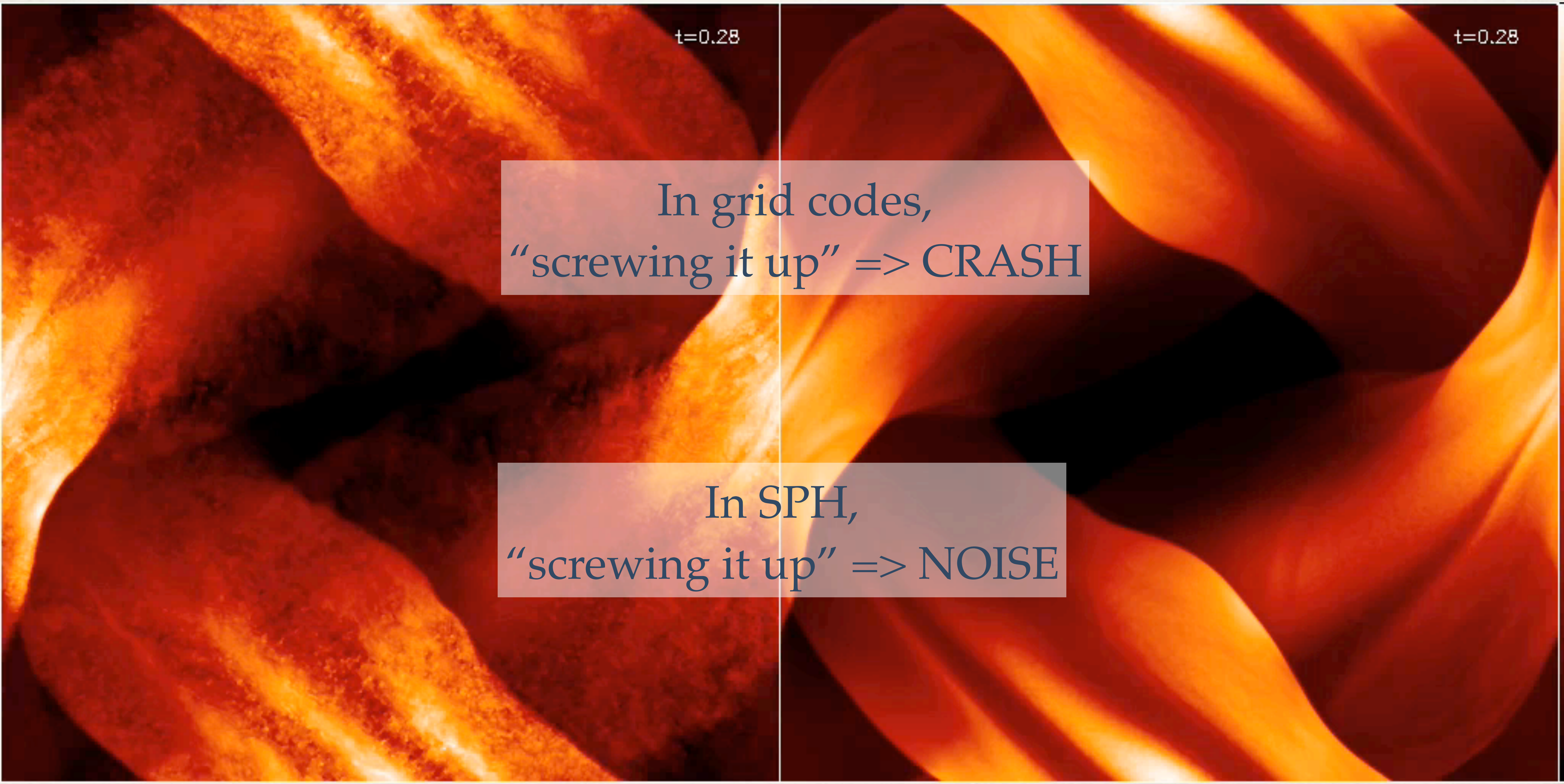
Exact conservation: Advantages



Orbits are orbits... even when they're not aligned with any symmetry axis.

Exact conservation: Disadvantages



Orszag-Tang Vortex in MHD (c.f. Price & Monaghan 2005, Rosswog & Price 2007, Price 2010)

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The minimum energy state

The “grid” in SPH…

SPH gradients 101

$$A_a = \sum_b \frac{m_b}{\rho_b} A_b W_{ab}$$

$$\nabla A_a = \sum_b \frac{m_b}{\rho_b} A_b \nabla W_{ab}$$

BAD

$$\nabla A_a = \sum_b \frac{m_b}{\rho_b} \left(A_b - A_a\right) \nabla W_{ab}$$

Exact constant

$$\chi_{\mu\nu} \nabla^\mu A_a = \sum_b m_b \left(A_b - A_a\right) \nabla W_{ab}$$

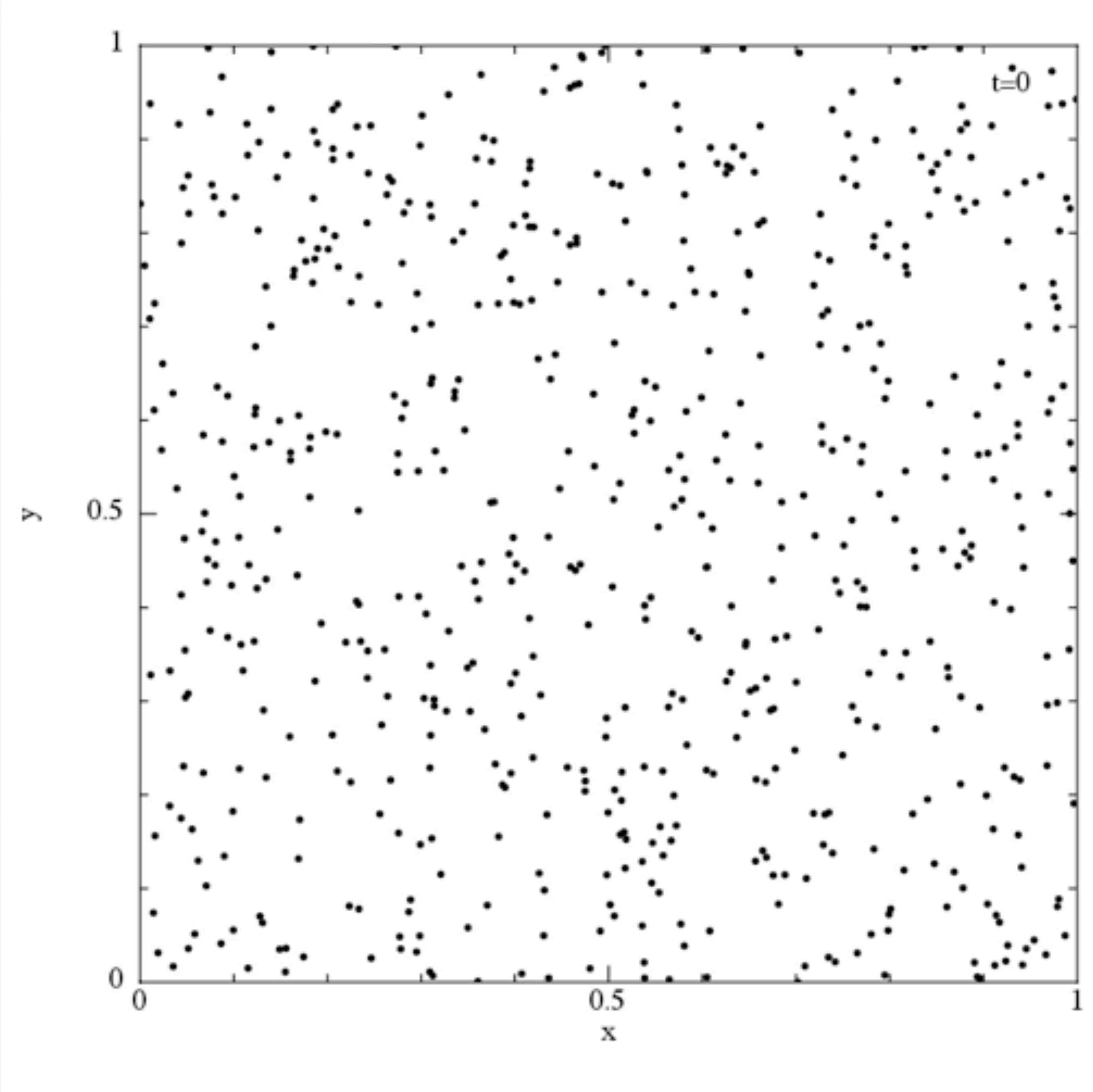
Exact linear

$$\chi_{\mu\nu} = \sum_b m_b (x^\mu - x^\mu) \nabla^\nu W_{ab}$$

$$\frac{\nabla A_a}{\rho_a} = -\sum_b m_b \left(\frac{A_a}{\rho_a^2} + \frac{A_b}{\rho_b^2} \right) \nabla W_{ab}$$

Huh?

What happens to a random particle arrangement?

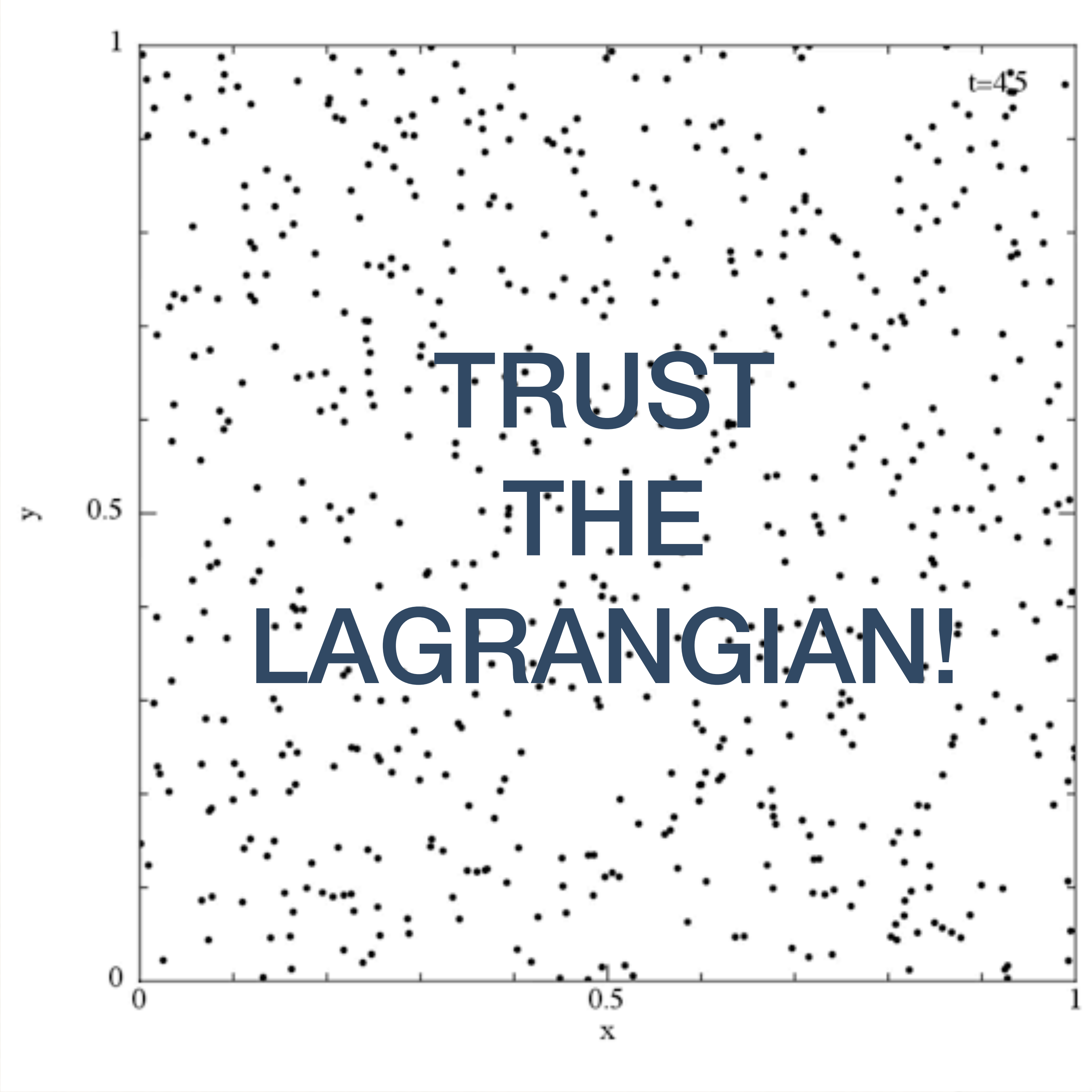


$$\frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla_i W_{ij}$$

SPH particles know how to stay regular

Why better gradients are a bad idea

Abel 2010, Price 2012

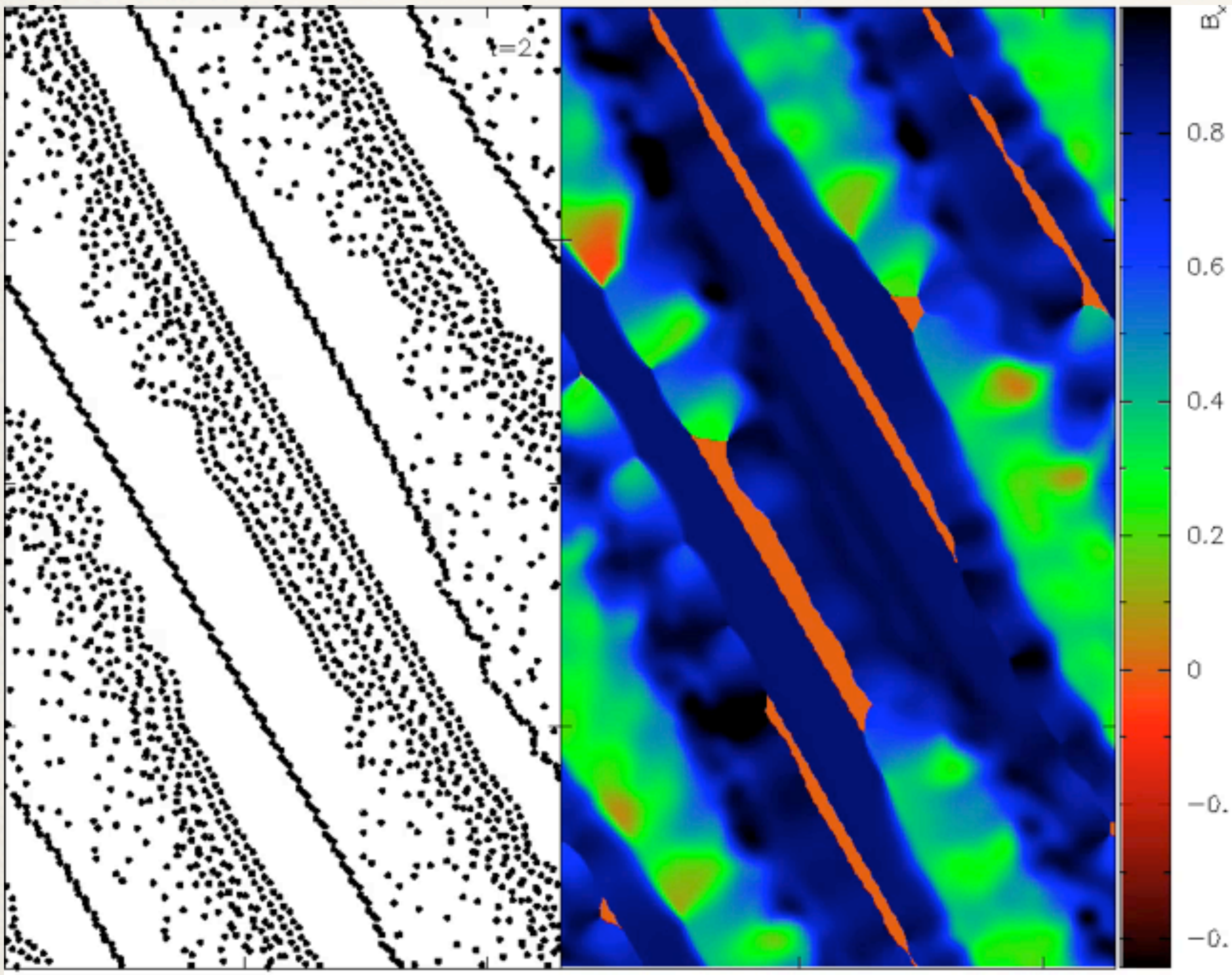


$$\frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = \sum_j m_j \left(\frac{P_i - P_j}{\rho_j^2} \right) \nabla_i W_{ij}$$

Improving the gradient operator leads to WORSE results

Corollary: Better to use a worse but conservative gradient operator

Corollary: Need positive pressures



MHD

$$S_{ij} = \left(P + \frac{B^2}{2\mu_0}\right)\delta_{ij} - \frac{B_i B_j}{\mu_0}$$

This is known as the tensile instability in SPH: occurs when net stress is negative

Smoothed Particle Magnetohydrodynamics

Price & Monaghan 2004a,b,2005, Price 2012

$$L_{sph} = \sum_b m_b \left[\frac{1}{2} v_b^2 - u_b(\rho_b, s_b) - \frac{1}{2\mu_0} \frac{B_b^2}{\rho_b} \right]$$

$$\int \delta L \mathrm{dt} = 0$$

$$\delta \rho_b = \sum_c m_c \left(\delta \mathbf{r}_b - \delta \mathbf{r}_c \right) \cdot \nabla_b W_{bc},$$

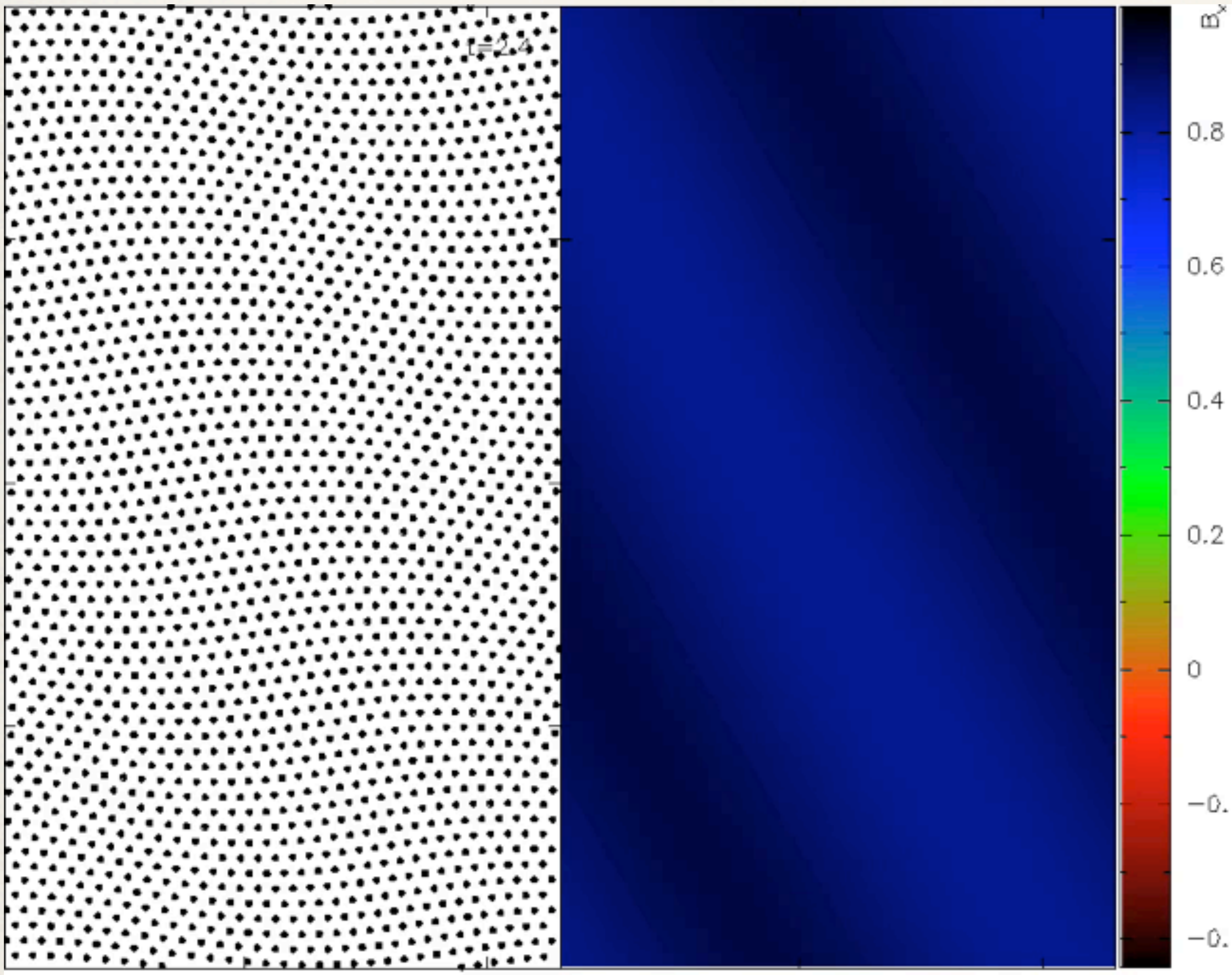
$$\delta \left(\frac{\mathbf{B}_b}{\rho_b} \right) = - \sum_c m_c (\delta \mathbf{r}_b - \delta \mathbf{r}_c) \frac{\mathbf{B}_b}{\rho_b^2} \cdot \nabla_b W_{bc}$$

$$\frac{dv_a^i}{dt} = - \sum_b m_b \left[\left(\frac{S^{ij}}{\rho^2} \right)_a + \left(\frac{S^{ij}}{\rho^2} \right)_b \right] \nabla_a^j W_{ab},$$

$$S_{ij} = \left(P + \frac{B^2}{2\mu_0} \right) \delta_{ij} - \frac{B_i B_j}{\mu_0}$$

Compromise approach gives stability

Børve, Omang & Trulsen (2001)



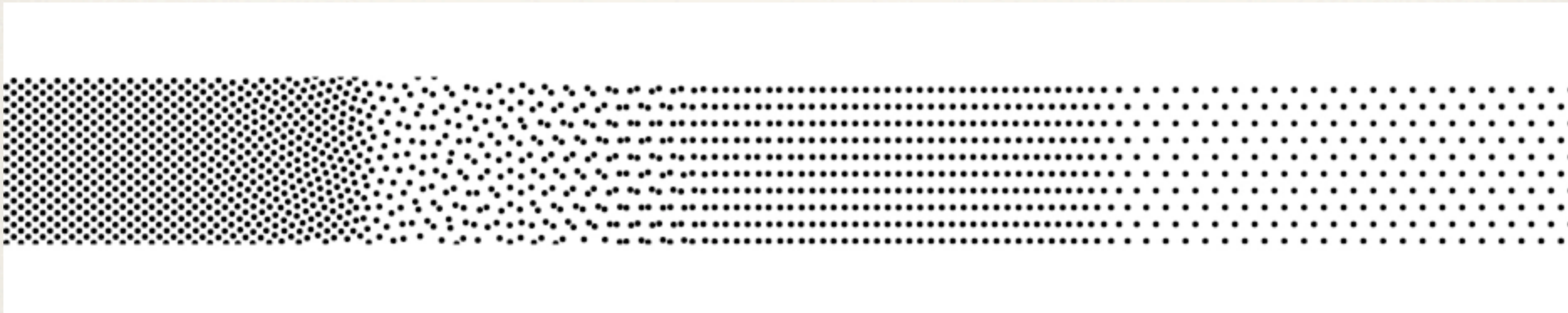
Subtract

$$-\mathbf{B}(\nabla\cdot\mathbf{B})$$

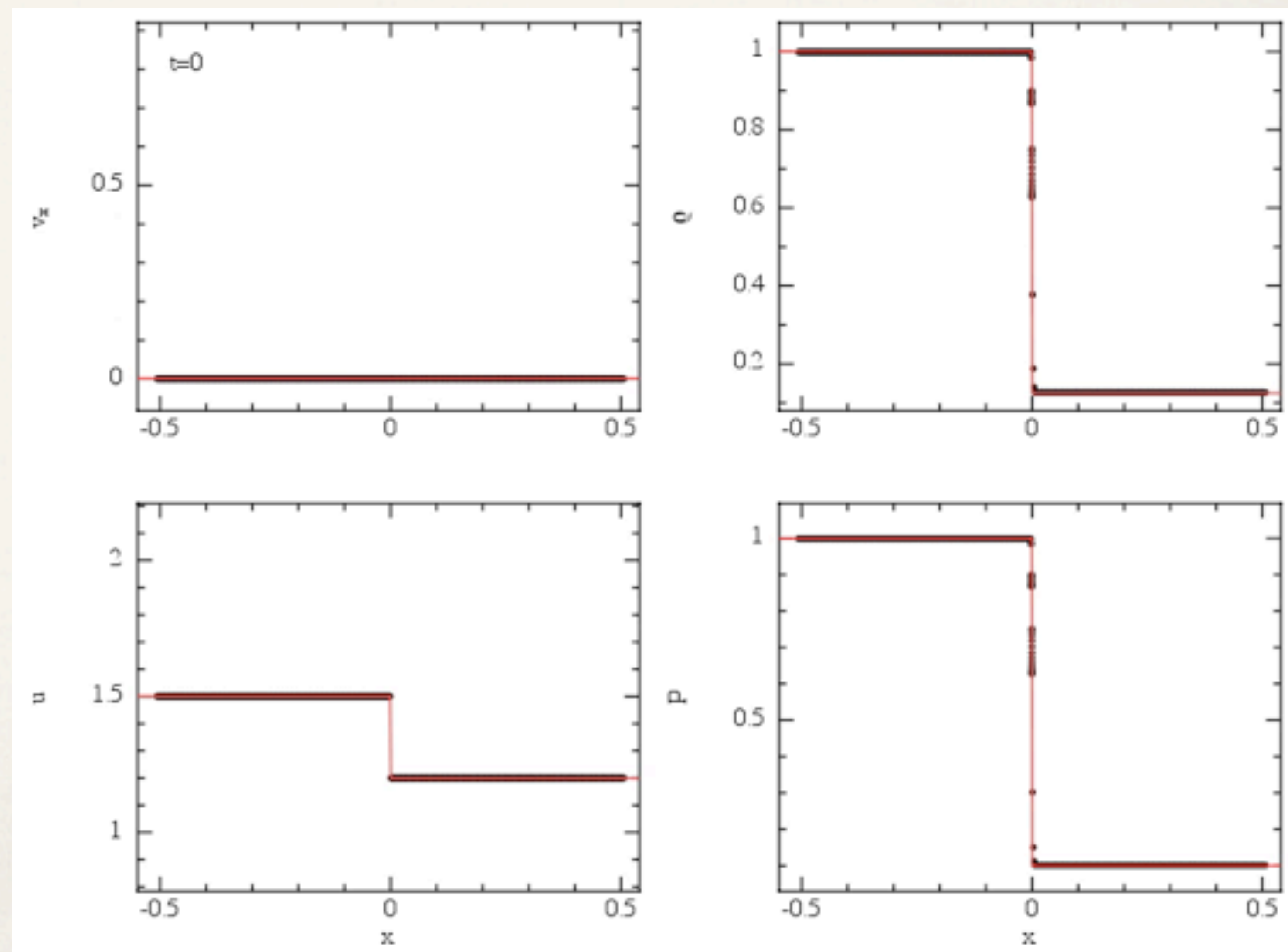
from MHD force:

Stable but nonconservative

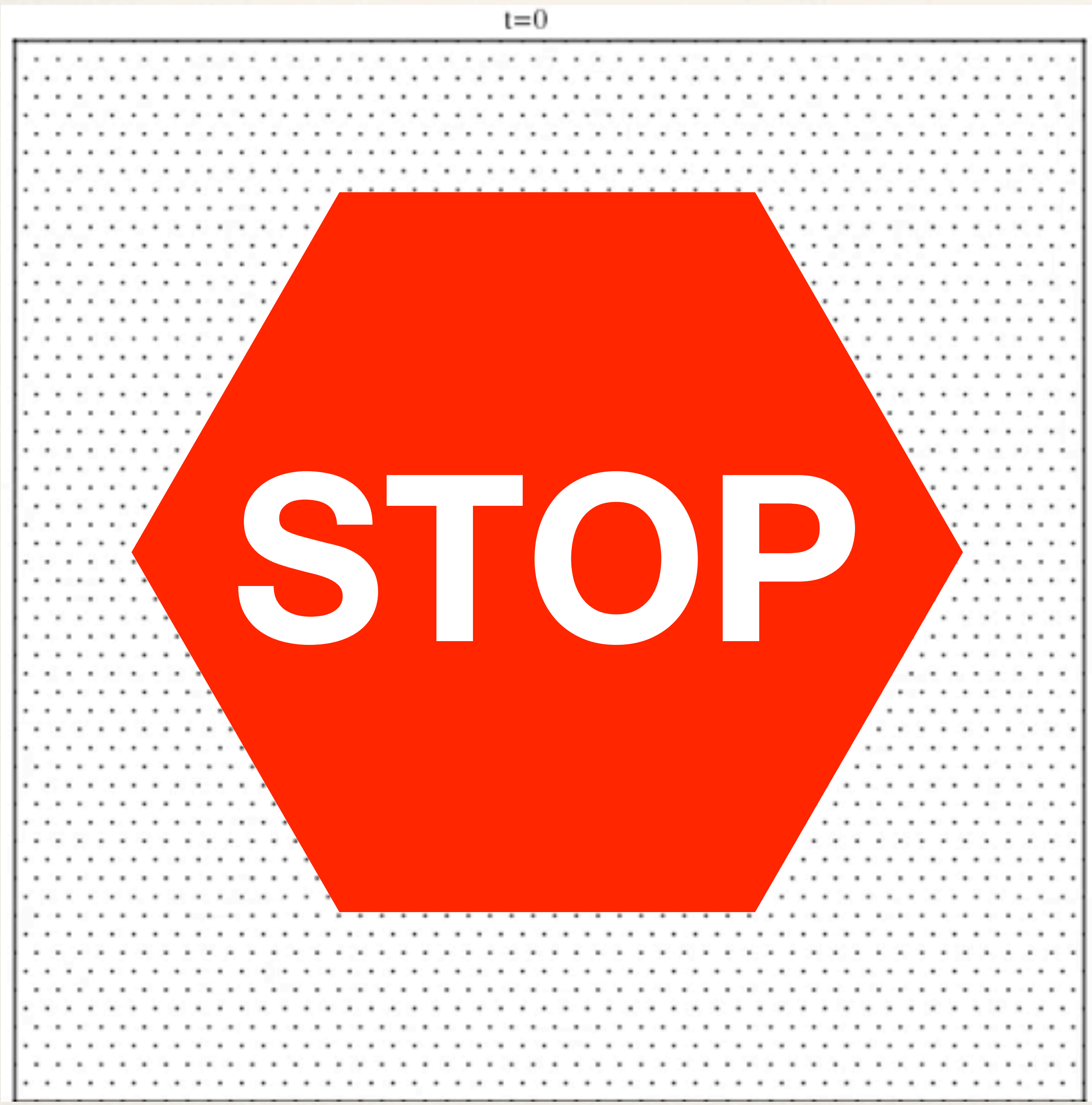
2D shock tube



* intrinsic “remeshing” of particles



Why you cannot use "more neighbours": The pairing instability



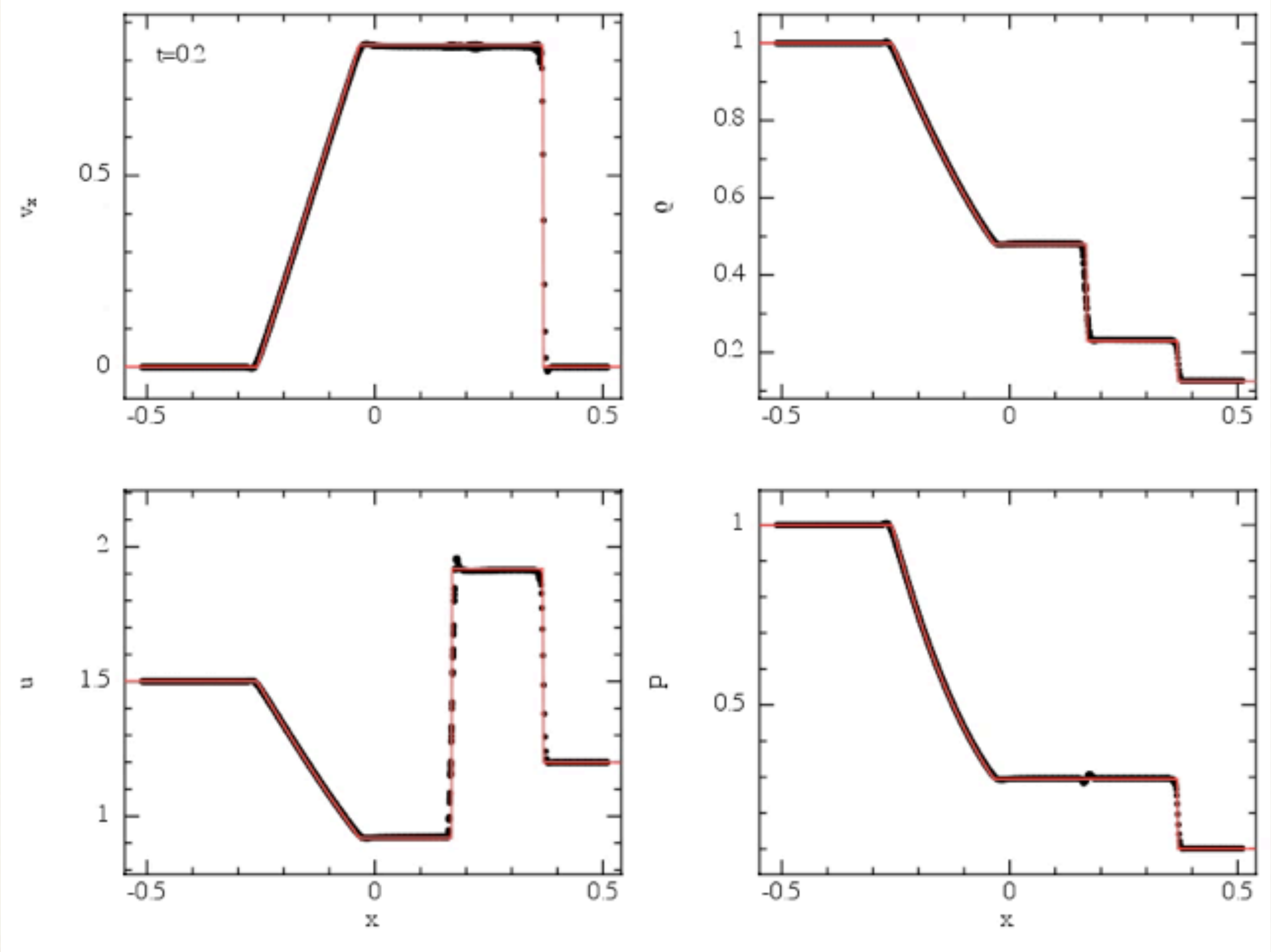
N_{neigh} should NOT be a free parameter!

i.e., cannot just increase the ratio of smoothing length to particle spacing with the B-spline kernels

pairing occurs for > 65 neighbours for the cubic spline in 3D

2D shock tube: M6 quintic

- use smoother M6 quintic kernel - truncated at 3h instead of 2h (NOT the same as "more neighbours" with the cubic spline)
- Resolution length given by different kernels scales with standard deviation of the kernel (Dehnen & Aly 2012, Leroy & Violeau 2013)



Pairing + Wendland kernels

Dehnen & Aly (2012)

- pairing depends on Fourier transform of the kernel
- Wendland Kernels: Fourier transform positive definite, hence no pairing, but are always biased
- B-splines: Fourier transform changes sign, pairing occurs at large neighbour number, but errors much smaller than Wendland for same number of neighbours

How to stop worrying and love Lagrangians

From density to hydrodynamics

$$L_{sph} = \sum_j m_j \left[\frac{1}{2}v_j^2 - u_j(\rho_j, s_j)\right]$$

Lagrangian

ADD PHYSICS

$+$

$$\mathrm{d}u = \frac{P}{\rho^2}\mathrm{d}\rho$$

1st law of thermodynamics

CHANGE SOMETHING HERE

$+$

$$\nabla\rho_i = \sum_j m_j \nabla W_{ij}(h)$$

density sum

$+$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{r}} = 0$$

Euler-Lagrange equations

$=$

$$\frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2}\right)\nabla_i W_{ij}(h)$$

equations of motion!

$$\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\nabla P}{\rho}\right)$$

Example I: Variable h

Springel & Hernquist 2002, Monaghan 2002, Price & Monaghan 2004b, 2007

$$\rho_a = \sum_b m_b W(\mathbf{r}_a - \mathbf{r}_b, h_a)$$

$$h_a = \eta \left(\frac{m_a}{\rho_a} \right)^{1/n_{dim}}$$

- nonlinear equation for h, rho
- requires iterative solution
- can solve to arbitrary precision

Euler-Lagrange equations

$$\frac{d\mathbf{v}_a}{dt} = -\sum_b m_b \left[\frac{P_a}{\Omega_a \rho_a^2} \nabla_a W_{ab}(h_a) + \frac{P_b}{\Omega_b \rho_b^2} \nabla_a W_{ab}(h_b) \right]$$

$$\Omega_a = \left[1 - \frac{dh_a}{d\rho_a} \sum_c m_c \frac{\partial W_{ab}(h_a)}{\partial h_a} \right]$$

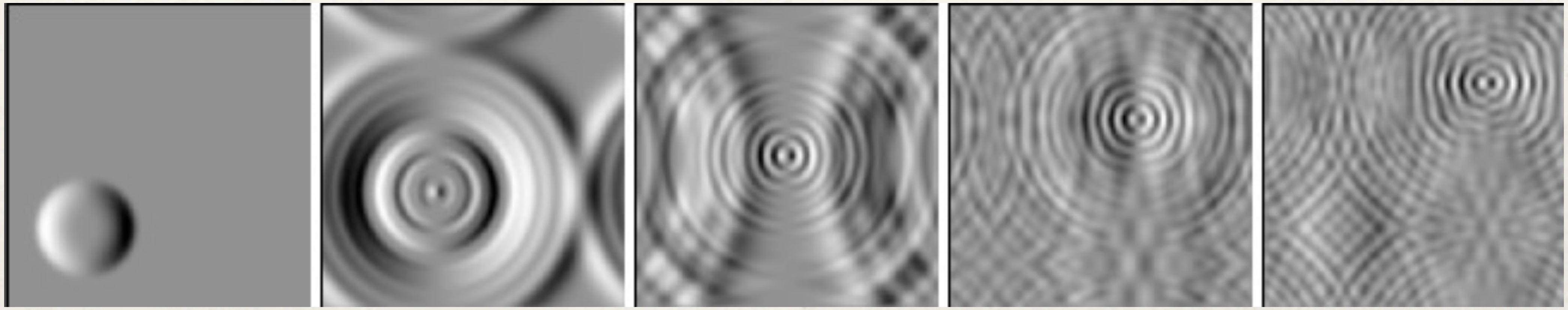
Example II: Hyperbolic divergence cleaning

Dedner et al. 2002, Price & Monaghan 2005, Tricco & Price 2012

$$\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)_{clean} = -\nabla\psi$$

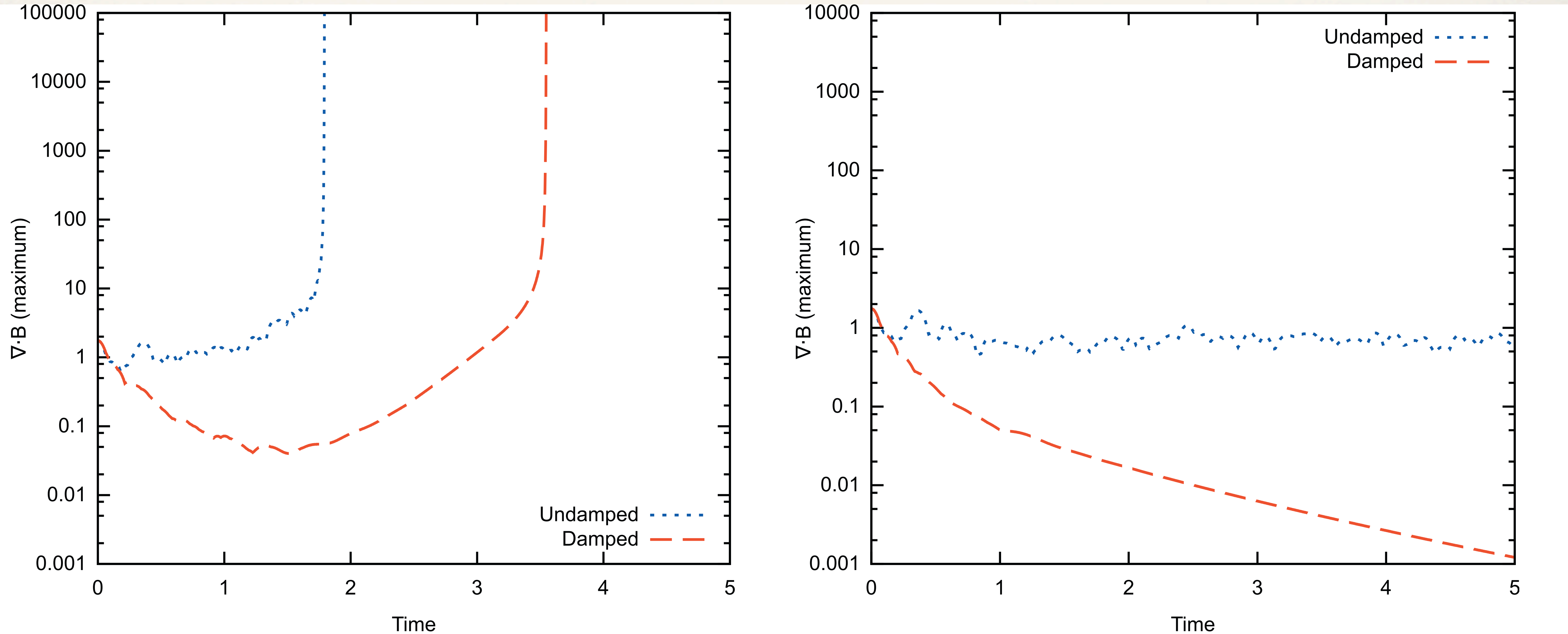
$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -c^2(\nabla\cdot\mathbf{B}) - \frac{\psi}{\tau}$$

these combine to give diffusive wave equation
for propagation of divergence errors:



Divergence cleaning (not from Lagrangian)

7224 *T.S. Tricco, D.J. Price / Journal of Computational Physics 231 (2012) 7214–7236*



Example II: Divergence cleaning

Price & Monaghan 2005, Tricco & Price 2012

$$L_{sph} = \sum_b m_b \left[\frac{1}{2}v_b^2 - u_b(\rho_b, s_b) - \frac{1}{2\mu_0}\frac{B_b^2}{\rho_b} - \frac{\psi_b^2}{2\mu_0 \rho_b c_b^2}\right]$$

$$\int \delta L \mathrm{dt} = 0$$

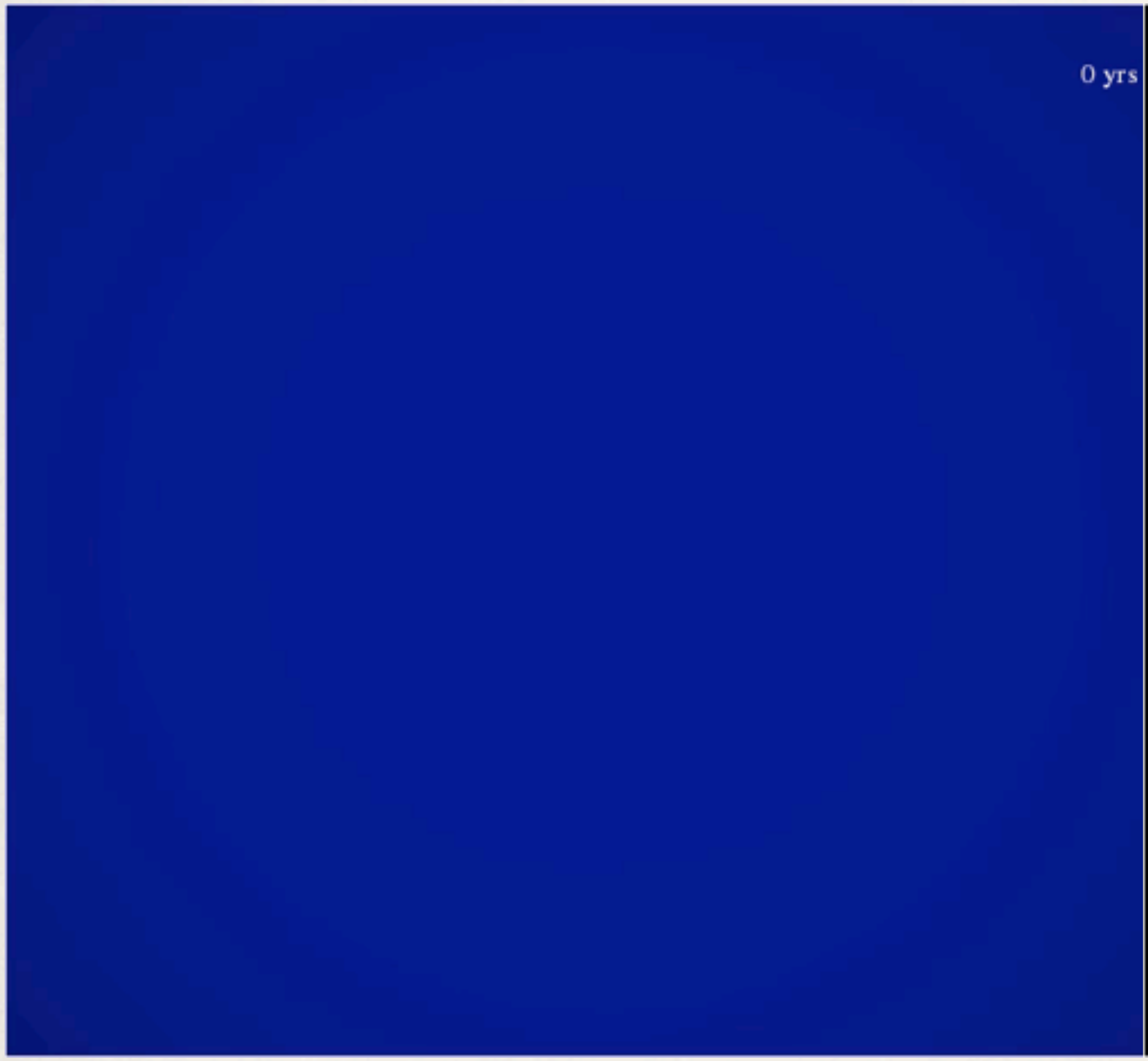
ADD PHYSICS

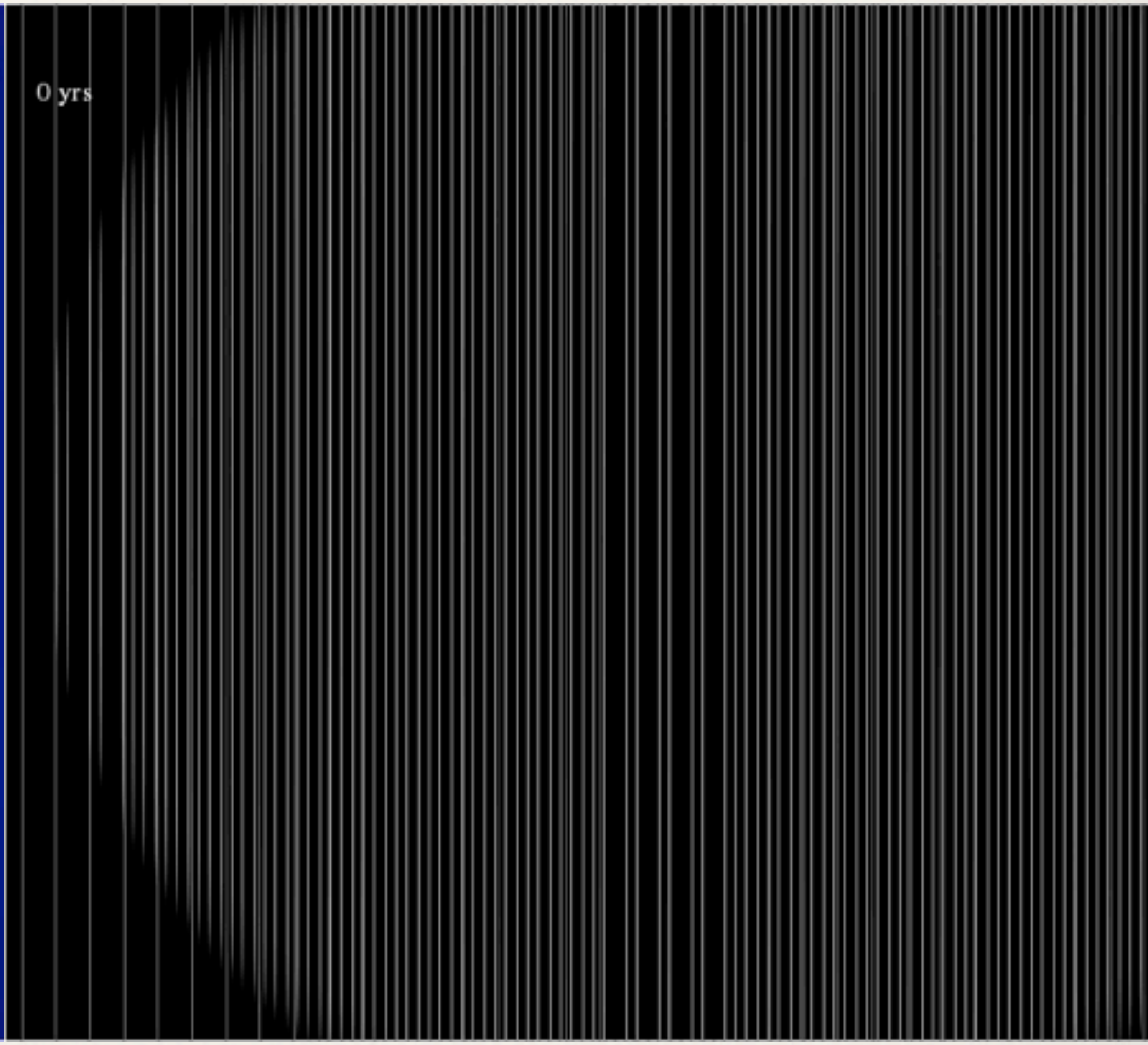
$$\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)_{clean} = -\rho_a \sum_b m_b \left[\frac{\psi_a}{\rho_a^2 \Omega_a}\nabla_a W_{ab}(h_a) + \frac{\psi_b}{\rho_b^2 \Omega_b}\nabla_a W_{ab}(h_b)\right]$$

$$\frac{\mathrm{d}\psi_a}{\mathrm{d}t} = -\frac{c_a^2}{\Omega_a \rho_a}\sum_b m_b (\mathbf{B}_b - \mathbf{B}_a)\cdot \nabla_a W_{ab}(h_a) - \frac{\psi_a}{\tau_a}$$

Magnetic jets from young stars

Tricco & Price 2012, Price, Tricco & Bate 2012





this explodes without divergence cleaning!

Conclusions

- Lagrangian approach gives a powerful way of both deriving and understanding SPH formulations
- Both advantages and disadvantages of SPH can be understood in this context

Summary: Advantages and disadvantages of SPH

Advantages:

- Resolution follows mass
- Zero dissipation until explicitly added
- Exact and simultaneous conservation of all physical quantities is possible
- Intrinsic remeshing procedure
- Does not crash

Disadvantages:

- Resolution follows mass
- Dissipation terms must be explicitly added to treat discontinuities
- Exact conservation means linear errors are worse
- Need to be careful with effects from particle remeshing
- Does not crash