Numerical Stability of SPH for Weakly Compressible Viscous Flows: Optimal Time-Stepping

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Outline

- Brief recall of the WCSPH tools
- SPH Von Neumann stability analysis
- Validation against numerical tests
- Model variations and applications to real flows
- Conclusions and recommendations

Time-stepping for WCSPH (1)

- The Lagrangian nature of SPH enhances **numerical instabilities**
- One of the most important **stability conditions** requires the time step to be bounded: $\delta t \leq \delta t_{crit}$

Time-stepping for WCSPH (1)

- The Lagrangian nature of SPH enhances **numerical instabilities**
- One of the most important **stability conditions** requires the time step to be bounded: $\delta t \leq \delta t_{crit}$
- The **critical time step** δt_{crit} should depend on the numerical parameters:
  - fluid reference density ρ_0
  - fluid (or numerical) kinematic viscosity ν
  - numerical speed of sound c_0
  - smoothing length h
- Thus, **dimensional analysis** gives $\delta t_{crit} = \frac{h}{c_0}\phi(\frac{c_0 h}{\nu})$
- ρ_0 has been removed as the only parameter depending on mass

Time-stepping for WCSPH (2)

- Notation:
  - **CFL number**: $C \doteq \frac{c_0 \delta t}{h}$
  - **Fourier number**: $C_\nu \doteq \frac{\nu \delta t}{h^2}$
  - Numerical **Reynolds number**: $Re_0 \doteq \frac{c_0 h}{\nu} = \frac{C}{C_\nu}$
- Thus, the stability condition reads $C \leq \phi(Re_0)$ or $C \leq \psi(C_\nu)$

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- *e.g.* Morris *et al.*, 1997 suggest **two empirical** conditions:
  - **Acoustic** condition: $C \leq 0.4$
  - **Viscous** condition: $C_\nu \leq 0.125$
  - ... or $C \leq min(0.4; 0.125 Re_0)$

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  - ... or $C \leq min(0.4; 0.125 Re_0)$
- The present work aims at deriving a **theoretical** time-stepping condition for WCSPH, *i.e.* a theoretical function ψ

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SPH gradient operators

- Basic **continuous** SPH gradient (no wall effects!):

$$
\begin{aligned}
\nabla A(\mathbf{r}) &\approx \int_{\Omega} \nabla A(\mathbf{r}')\, w_h(|\mathbf{r}-\mathbf{r}'|)\, d\mathbf{r}' \\
&= \cancel{\int_{\partial\Omega} A(\mathbf{r}')\, w_h(|\mathbf{r}-\mathbf{r}'|)\, \mathbf{n}' d\Gamma'} + \int_{\Omega} A(\mathbf{r}')\, \nabla_{\mathbf{r}} w_h(|\mathbf{r}-\mathbf{r}'|)\, d\mathbf{r}'
\end{aligned}
$$

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\begin{aligned}
\nabla A(\mathbf{r}) &= \rho(\mathbf{r})\, \nabla \frac{A}{\rho}(\mathbf{r}) + \frac{A}{\rho}(\mathbf{r})\, \nabla\rho(\mathbf{r}) \\
&= \int_{\Omega} \frac{\rho(\mathbf{r}')^2 A(\mathbf{r}) + \rho(\mathbf{r})^2 A(\mathbf{r}')}{\rho(\mathbf{r})\, \rho(\mathbf{r}')} \nabla_{\mathbf{r}} w_h(|\mathbf{r}-\mathbf{r}'|)\, d\mathbf{r}' \doteqdot \mathbf{G}^{+}\{A\}(\mathbf{r})
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$$

- **Discrete** SPH gradient:

$$
\mathbf{G}_a^{+}\{A_b\} \doteqdot \rho_a \sum_b m_b \left(\frac{A_a}{\rho_a^2} + \frac{A_b}{\rho_b^2} \right) \nabla w_{ab}
$$

Other SPH operators

- SPH divergence:

$$
\begin{aligned}
D^{-}\left\{\mathbf{A}\right\}\left(\mathbf{r}\right) &\doteq \int_{\Omega} \frac{\rho\left(\mathbf{r}'\right)}{\rho\left(\mathbf{r}\right)} \left[\mathbf{A}\left(\mathbf{r}'\right) - \mathbf{A}\left(\mathbf{r}\right)\right] \cdot \nabla_{\mathbf{r}} w_h\left(\left|\mathbf{r} - \mathbf{r}'\right|\right) d\mathbf{r}' \\
D_a^{-}\left\{\mathbf{A}_b\right\} &\doteq \frac{1}{\rho_a} \sum_b m_b \left(\mathbf{A}_b - \mathbf{A}_a\right) \cdot \nabla w_{ab}
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$$D_a^{-}\left\{\mathbf{A}_b\right\} \doteqdot \frac{1}{\rho_a}\sum_b m_b\left(\mathbf{A}_b-\mathbf{A}_a\right)\cdot\nabla w_{ab}$$

- SPH Laplacian:

$$\mathbf{L}\left\{\mathbf{A}\right\}\left(\mathbf{r}\right) \doteqdot 2\int_{\Omega}\left[\mathbf{A}\left(\mathbf{r}\right)-\mathbf{A}\left(\mathbf{r}'\right)\right]\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^2}\cdot\nabla_{\mathbf{r}} w_h\left(\left|\mathbf{r}-\mathbf{r}'\right|\right) d\mathbf{r}'$$

$$\mathbf{L}_a\left\{\mathbf{A}_b\right\} \doteqdot 2\sum_b V_b\left(\mathbf{A}_a-\mathbf{A}_b\right)\frac{\mathbf{r}_{ab}}{r_{ab}^2}\cdot\nabla w_{ab}$$

  - Other formulae exist (**see later**)
  - Complete formulae involve **boudary terms** (no wall effects here!)
  - Discrete operators should be **renormalized** for consistency

Standard WCSPH model

- Discrete form of the Lagrangian **Navier-Stokes** equations:

$$\begin{aligned}
\dot{\mathbf{u}}_a &= -\frac{1}{\rho_a}\mathbf{G}_a^{+}\left\{p_b\right\} + \frac{\mu}{\rho_a}\mathbf{L}_a\left\{\mathbf{u}_b\right\} \\
\dot{\mathbf{r}}_a &= \mathbf{u}_a \\
\dot{\rho}_a &= -\rho_a D_a^{-}\left\{\mathbf{u}_b\right\} \\
p_a &= \frac{\rho_0 c_0^2}{\gamma}\left(\frac{\rho_a^{\gamma}}{\rho_0^{\gamma}} - 1\right)
\end{aligned}$$

- Definitions:
  - $\mu \doteqdot \rho_0 \nu$: dynamic viscosity
  - c_0 is set as $10 U_{max}$ to ensure weakly compressible flow
  - $\gamma = 7$ for water (Monaghan, 1994)
- Note: a **time marching scheme** is also required (see later)

Monaghan, J.J. (1994), J. Comput. Phys. **110**:399–406

Von Neumann stability analysis

- Principles of a **von Neumann** stability analysis:
  - Writing the governing equations $\dot{\mathbf{X}} = \mathbf{g}\left(\mathbf{X}\right)$
  - Identifying a reference state $\mathbf{X}_{ref}$ satisfying $\dot{\mathbf{X}}_{ref} = \mathbf{g}\left(\mathbf{X}_{ref}\right)$
  - Searching a perturbated solution $\mathbf{X} = \mathbf{X}_{ref} + \delta\mathbf{X}$ by linearizing:

$$\delta\dot{\mathbf{X}} = \delta\mathbf{g}\left(\mathbf{X}\right) = \left(\frac{\partial\mathbf{g}}{\partial\mathbf{X}}\right)_{\mathbf{X}=\mathbf{X}_{ref}} \delta\mathbf{X}$$

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  - Searching wave-like solutions: $\delta\mathbf{X} = \mathbf{X}_0 e^{-i\mathbf{K}\cdot\mathbf{r}+i\omega t}$
  - The linearized system gives a dispersion relation, *i.e.* a relation between the wave vector $\mathbf{K}$ and the angular frequency $\omega = \omega\left(\mathbf{K}\right)$

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  - The linearized system gives a dispersion relation, *i.e.* a relation between the wave vector $\mathbf{K}$ and the angular frequency $\omega = \omega\left(\mathbf{K}\right)$
- Stability criteria:
  - **Physical equations** (continuous time): $\forall\mathbf{K}, \operatorname{Im}\omega \geq 0$
  - **Numerical model** (discrete time): $\forall\mathbf{K}$, $|\chi| \leq 1$, where $\chi \doteqdot e^{i\omega\delta t}$ is the (numerical) wave amplification factor

Linearization of the SPH equations

- $\mathbf{X}$ represents the set of all particle parameters $\mathbf{u}_a$, $\mathbf{r}_a$ and ρ_a
- Possible reference state: constant velocity and density, *i.e.* we search $\mathbf{u}_a = \mathbf{u}_{ref} + \delta\mathbf{u}_a$, $\mathbf{r}_a = \mathbf{r}_{a,ref} + \delta\mathbf{r}_a$, $\rho_a = \rho_{ref} + \delta\rho_a$:

$$\delta\left[\rho_a D_a^- \{\mathbf{u}_b\}\right] = \delta\left[\sum_b m_b\left(\mathbf{u}_b - \mathbf{u}_a\right)\cdot\nabla w_{ab}\right]$$

$$= \sum_b m_b\left[\left(\delta\mathbf{u}_b - \delta\mathbf{u}_a\right)\cdot\nabla w_{ab} + \cancel{\left(\mathbf{u}_{ref} - \mathbf{u}_{ref}\right)\cdot\nabla\nabla w_{ab}\left(\delta\mathbf{r}_a - \delta\mathbf{r}_b\right)}\right]$$

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- The last term vanishes, so:

$$\begin{aligned}\delta\dot{\rho}_a &= -\delta\left[\rho_a D_a^- \left\{\mathbf{u}_b\right\}\right] \\ &\approx \rho_0\int_\Omega\left[\delta\mathbf{u_a} - \delta\mathbf{u}\left(\mathbf{r}'\right)\right]\cdot\nabla_{\mathbf{r_a}} w_h\left(\left|\mathbf{r_a} - \mathbf{r}'\right|\right)d\mathbf{r}'\end{aligned}$$

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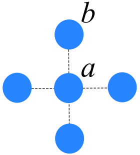
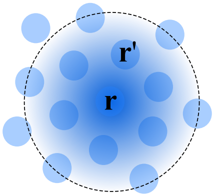
- Note: starting from **continuous** SPH would be easier!

Discrete or continuous?

- The stability of SPH can be studied from two ways:
    - **Discrete**: Cartesian grid, one neighbour in each direction
    - **Continuous**: ignores the discrete nature of SPH

Space dimension n	**1**	**arbitrary**
Discrete	Swegle *et al.*, 1995 Morris, 1996	De Leffe, 2011 Dehnen & Aly, 2012
Continuous	Balsara, 1995	Dehnen & Aly, 2012 Present work





Swegle, J.W., Hicks, D.L., Attaway, S.W. (1995), J. Comput. Phys. **116**:123–134
Morris, J.P. (1996), Ph.D. thesis, Melbourne
De Leffe, M. (2011), Ph.D. thesis, Ecole Centrale de Nantes (in French)
Balsara, D.S. (1995), J. Comput. Phys. **121**:357–372
Dehnen, W., Aly, H. (2012), Mon. Not. R. Astron. Soc. **000**:1–15

Solutions in the Fourier space

- We now search solutions as $\delta \mathbf{u}_a = c_0 \mathbf{U}(t) e^{-i\mathbf{K}\cdot\mathbf{r_a}}$, $\mathbf{K}$ being a numerical wave vector:

$$\delta \dot{\rho}_a = \rho_0 c_0 \mathbf{U}(t) \cdot \int_\Omega \left(e^{-i\mathbf{K}\cdot\mathbf{r_a}} - e^{-i\mathbf{K}\cdot\mathbf{r}'} \right) \nabla_{\mathbf{r_a}} w_h \left(|\mathbf{r_a} - \mathbf{r}'| \right) d\mathbf{r}'$$

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- With the variable change $\tilde{\mathbf{r}} \doteqdot \mathbf{r}' - \mathbf{r_a}$, *i.e.* $\nabla_{\mathbf{r_a}} = -\nabla_{\tilde{\mathbf{r}}}$:

$$\begin{aligned} e^{i\mathbf{K}\cdot\mathbf{r_a}}\delta\dot{\rho}_a &= \rho_0 c_0 \mathbf{U}\left(t\right)\cdot\int_\Omega \left(e^{-i\mathbf{K}\cdot\tilde{\mathbf{r}}} - 1\right)\nabla_{\tilde{\mathbf{r}}} w_h\left(\tilde{r}\right)d\tilde{\mathbf{r}} \\ &= \rho_0 c_0 \mathbf{U}\left(t\right)\cdot\widehat{\nabla_{\tilde{\mathbf{r}}} w_h}\left(K\right) \\ &= i\rho_0 c_0 \widehat{w_h}\left(K\right)\mathbf{K}\cdot\mathbf{U}\left(t\right) \end{aligned}$$

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- The **Fourier transform of the kernel** is thus important in studying the numerical stability properties of SPH.

Linearized WCSPH system

- Similarly to the velocity, positions and density are searched for in the following forms:
  - $\delta \mathbf{r}_a = h\mathbf{R}\left(t\right)e^{-i\mathbf{K}\cdot\mathbf{r_a}}$
  - $\delta \rho_a = \rho_0 R\left(t\right)e^{-i\mathbf{K}\cdot\mathbf{r_a}}$

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  - $\delta \rho_a = \rho_0 R\left(t\right) e^{-i\mathbf{K}\cdot\mathbf{r_a}}$
- After some algebra the linearized WCSPH system reads:

$$\begin{aligned}
\dot{\mathbf{U}}\left(t\right) &= \frac{ic_0}{h}\widehat{w_h}\left(K^+\right) R\left(t\right)\mathbf{K}^+ - \frac{\nu}{h^2}F_2\left(K^+\right)\mathbf{U}\left(t\right) \\
\dot{\mathbf{R}}\left(t\right) &= \frac{c_0}{h}\mathbf{U}\left(t\right) \\
\dot{R}\left(t\right) &= \frac{ic_0}{h}\widehat{w_h}\left(K^+\right)\mathbf{K}^+\cdot\mathbf{U}\left(t\right)
\end{aligned}$$

- $\mathbf{K}^+ \doteq h\mathbf{K}$ is the dimensionless wavevector, $K^+ \doteq \left|\mathbf{K}^+\right|$ and

$$F_2\left(K^+\right) \doteq 2h^2\int_\Omega \left(e^{-i\mathbf{K}\cdot\tilde{\mathbf{r}}} - 1\right)\frac{\tilde{\mathbf{r}}}{\tilde{r}^2}\cdot\nabla_{\tilde{\mathbf{r}}} w_h\left(\tilde{r}\right) d\tilde{\mathbf{r}}$$

Time marching scheme

- We first consider a first order **semi-explicit scheme**:
  - Time derivatives are approximated as $\dot{\mathbf{U}}\left(t\right)=\frac{\mathbf{U}\left(t^{m+1}\right)-\mathbf{U}\left(t^{m}\right)}{\delta t}$
  - **Updated velocities** are used to compute positions and densities
  - We search all functions of time as $\mathbf{U}\left(t\right)=\mathbf{U}_0e^{i\omega t}$, etc.

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  - **Updated velocities** are used to compute positions and densities
  - We search all functions of time as $\mathbf{U}\left(t\right) = \mathbf{U}_0 e^{i\omega t}$, etc.
- The linear system now reads:

$$\begin{aligned} \frac{\chi - 1}{\delta t}\mathbf{U}_0 &= \frac{ic_0}{h}\widehat{w_h}\left(K^+\right)R_0\mathbf{K}^+ - \frac{\nu}{h^2}F_2\left(K^+\right)\mathbf{U}_0 \\ \frac{\chi - 1}{\delta t}\mathbf{R}_0 &= \chi\frac{c_0}{h}\mathbf{U}_0 \\ \frac{\chi - 1}{\delta t}R_0 &= \chi\frac{ic_0}{h}\widehat{w_h}\left(K^+\right)\mathbf{K}^+ \cdot \mathbf{U}_0 \end{aligned}$$

- Recall $\chi \doteqdot e^{i\omega\delta t}$ is the numerical **wave amplification factor**. The stability condition reads $\forall\mathbf{K}^+, |\chi| \leq 1$.

The eigenvalue problem

- Rearranging the system leads to an **eigenvalue problem**:

$$\chi A_1 \left(\mathbf{K}^+ \otimes \mathbf{K}^+\right) \mathbf{U}_0 = -\left(\chi - 1 + A_2\right)\left(\chi - 1\right) K^{+2} \mathbf{U}_0$$

- New notation:

$$\begin{aligned} A_1 &\doteq C^2 F_1\left(K^+\right) \\ A_2 &\doteq C_\nu F_2\left(K^+\right) \\ F_1\left(K^+\right) &\doteq \left[K^+ \widehat{w_h}\left(K^+\right)\right]^2 \end{aligned}$$

- Recall:

$$C \doteq \frac{c_0 \delta t}{h} \quad C_\nu \doteq \frac{\nu \delta t}{h^2} = \frac{C}{Re_0}$$

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- Integration by parts (no wall effect!) gives:

$$F_2'\left(K^+\right) = 2K^+\widehat{w_h}\left(K^+\right) = 2\sqrt{F_1\left(K^+\right)}$$

- Recall:

$$C \doteqdot \frac{c_0 \delta t}{h} \quad C_\nu \doteqdot \frac{\nu \delta t}{h^2} = \frac{C}{Re_0}$$

Stability criterion

- The tensor $\mathbf{K}^{+} \otimes \mathbf{K}^{+}$ has two eigenvalues: 0 and K^{+2}
- Only the second is important to investigate. It gives the following **characteristic polynomial**:

$$\chi^2 + (A_1 + A_2 - 2)\,\chi + 1 - A_2 = 0$$

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$$\chi^2 + (A_1 + A_2 - 2)\,\chi + 1 - A_2 = 0$$

- The roots satisfy the stability criterion $\forall \mathbf{K}^+, |\chi| \leq 1$ if and only if $A_1 + 2A_2 \leq 4$ **for all wavenumbers**, *i.e.*:

$$C \leq \sqrt{2 \min_{K^+} \frac{2 - C_\nu F_2\left(K^+\right)}{F_1\left(K^+\right)}} = \psi(C_\nu)$$

- For comparison, recall Morris *et al.*'s 'traditional' empirical criteria: $C \leq 0.4$ and $C_\nu \leq 0.125$

The stability functions (1)

- Note: $\widehat{w_h}$, F_1 and F_2 depend on $K^+ \doteq |\mathbf{K}^+|$ only for **isotropy** reasons (no wall effect here!)
- **Kernel** notation:

$$w_h\left(\tilde{r}\right) = \frac{\alpha_n}{h^n} f\left(q\right) \qquad q \doteq \frac{\tilde{r}}{h}$$

α_n being a normalizing constant and n the space dimension.

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  α_n being a normalizing constant and n the space dimension.

- Example 1: the **Gaussian** kernel:

$$f(q) = e^{-q^2}$$

  with $\alpha_n = \pi^{-n/2}$.

$$\widehat{w_h}\left(K^+\right) = e^{-\frac{K^{+2}}{4}}$$

$$F_1\left(K^+\right) = K^{+2} e^{-\frac{K^{+2}}{2}} \qquad F_2\left(K^+\right) = 4\left(1 - e^{-\frac{K^{+2}}{4}}\right)$$

The stability functions (2)

- Example 2: the **Wendland** kernel or order 5:

$$f\left(q\right) = \left(1-\frac{q}{2}\right)^4\left(1+2q\right) \qquad \text{if } 0 \leq q \leq 2$$

with $\alpha_1 = 3/4$, $\alpha_2 = 7/4\pi$, $\alpha_3 = 21/16\pi$.

$$n = 1 : \widehat{w_h}\left(K^+\right) = \tfrac{45}{2K^{+6}}\left(K^{+2} + \tfrac{1}{2}K^+ \sin 2K^+ - 2\sin^2 K^+\right)$$

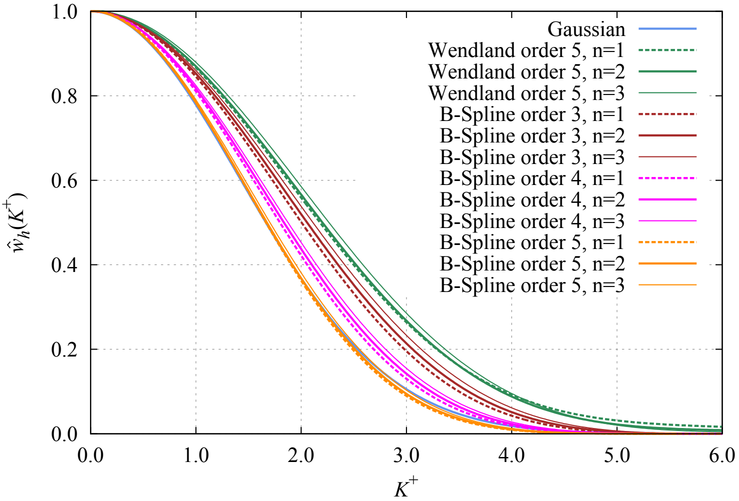
$$n = 2 : \widehat{w_h}\left(K^+\right) = \tfrac{105}{4K^{+6}}\left[\begin{array}{c} 6K^{+2}J_0\left(2K^+\right) - K^+J_1\left(2K^+\right) \\ +3\pi\left(K^{+2}-\tfrac{5}{4}\right)Y\left(2K^+\right) \end{array}\right]$$

$$n = 3 : \widehat{w_h}\left(K^+\right) = \tfrac{315}{8K^{+8}}\left[\begin{array}{c} \left(12-2K^{+2}\right)\cos 2K^+ \\ +9K^+\sin 2K^+ + 8K^{+2} - 12 \end{array}\right]$$

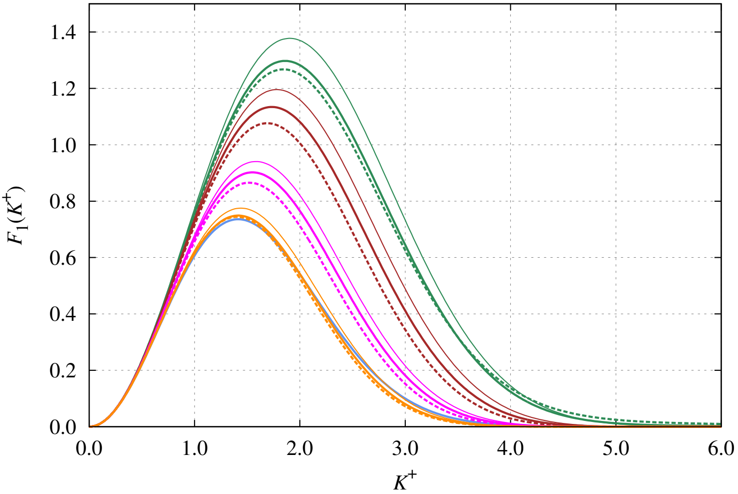
$$Y\left(x\right) \doteqdot J_1\left(x\right)H_0\left(x\right) - J_0\left(x\right)H_1\left(x\right)$$

where J_0, J_1 are Bessel functions and H_0, H_1 Struve functions (Abramovic and Stegun, 1972).

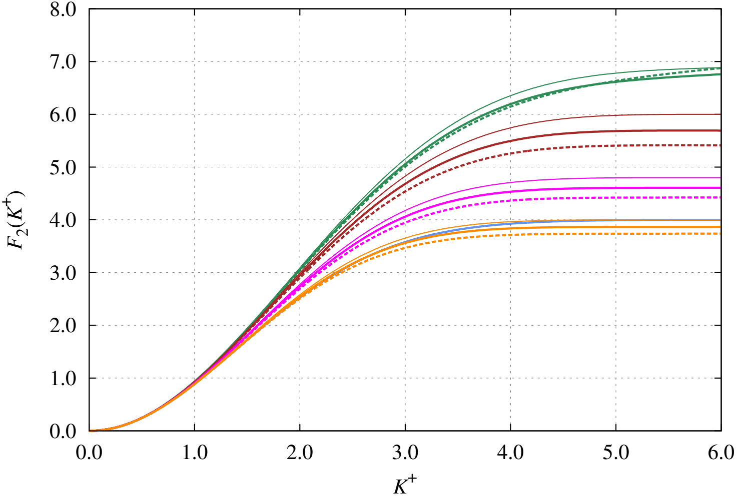
Function $\widehat{w_h}\left(K^+\right)$



Function $F_1\left(K^+\right)$

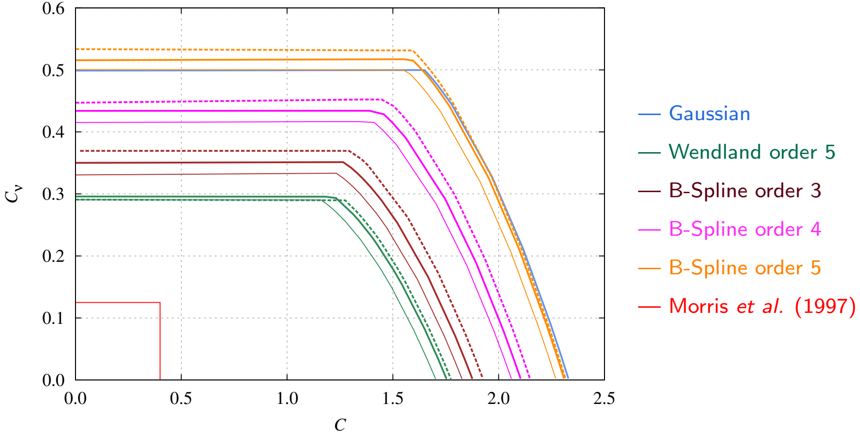


Function $F_2\left(K^+\right)$



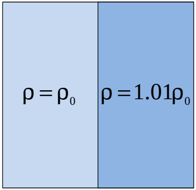
Stability domains

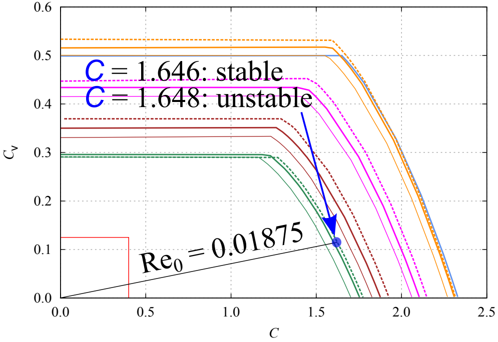
$$C \leq \sqrt{2 \min_{K^+} \frac{2 - C_\nu F_2\left(K^+\right)}{F_1\left(K^+\right)}}$$



Numerical validation

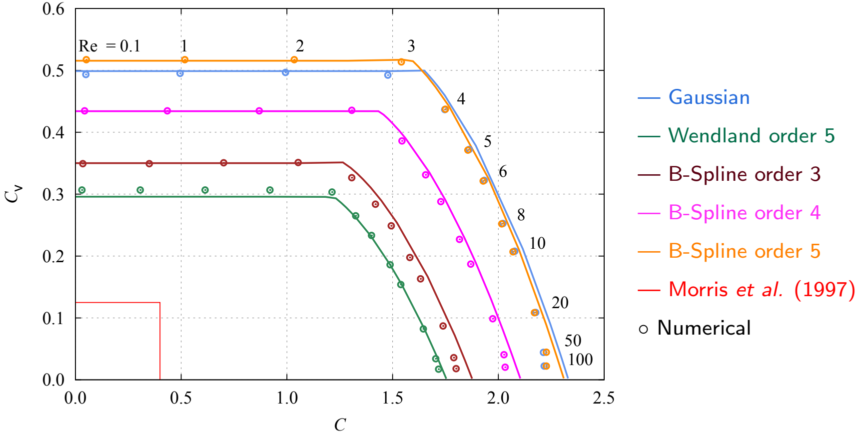
- The '**infinite flow**' test case
    - $n = 2$, square of 40×40 particles
    - Double periodicity ('infinite flow')
    - $\mathbf{u}_{ref} = \mathbf{0}$ by Galilean invariance
    - 1 % initial density discontinuity





Stability domains: validation

- 'Infinite flow' test case in dimension $n = 2$



Maximum Reynolds number Re_0

- Numerically, $Re_0 \doteq \frac{c_0 h}{\nu}$ could not exceed a **critical value** $Re_{crit} \sim 100$. This may be due to:
  - The discrete nature of SPH (tensile instability, see Swegle *et al.*, 1995), not explained by the present theory
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  - By chance, with $c_0 \sim 10 U_{max}$, $Re_0 \sim 100 \Longleftrightarrow Re \sim 100$ to 1000 (according to the space resolution $\frac{L}{h}$)
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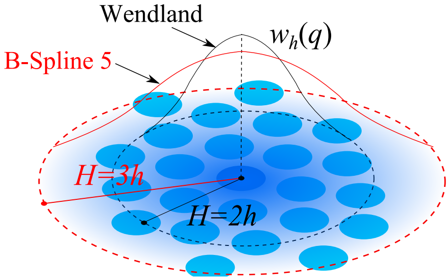
  - However, the SPH instability at large Re_0 is not representative of physical turbulence growth
- **Solutions** to keep Re_0 below ~ 100 (c_0 cannot be decreased)
  - Decreasing h (finer space resolution): DNS
  - Increasing ν: RANS model with eddy viscosity closure

Re-scaling the kernels (1)

- The size of the **kernel support** is not only determined by h
- Dehnen and Aly, 2012 suggest to use as a measure of space resolution the **kernel standard deviation** σ in place of h:

$$\sigma^2 \doteqdot \frac{1}{n} \int_{\Omega} \tilde{r}^2 w_h(\tilde{r})\, d\tilde{\mathbf{r}}$$

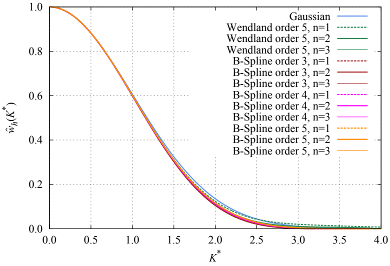


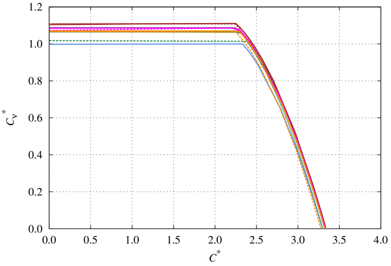
Dehnen, W., Aly, H. (2012), Mon. Not. R. Astron. Soc. **000**:1–15

Re-scaling the kernels (2)

- $K^* \doteqdot \sigma K$ should now be used in place of $K^+ \doteqdot hK$
- The **re-scaled** kernel Fourier transforms $\widehat{w_h}(K^*)$ come much closer together, as well as $F_1(K^*)$ and $F_2(K^*)$
- As a consequence, so do the stability domains, with the new definitions:

$$C^* \doteqdot \frac{c_0 \delta t}{\sigma} \qquad C_\nu^* \doteqdot \frac{\nu \delta t}{\sigma^2}$$





Model variations (1)

- **Density interpolation** instead of continuity equation:

$$\rho_a = \sum_b m_b w_{ab}$$

  - The theoretical stability domain is **unchanged**
  - This is confirmed by numerical tests

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- **Modified** gradient and divergence **operators**:

$$\begin{aligned} \mathbf{G}_a^k \left\{A_b\right\} &\doteqdot \sum_b V_b \frac{\rho_b^{2k} A_a + \rho_a^{2k} A_b}{\left(\rho_a \rho_b\right)^k} \nabla w_{ab} \\ D_a^k \left\{\mathbf{A}_b\right\} &\doteqdot -\frac{1}{\rho_a^{2k}} \sum_b V_b \left(\rho_a \rho_b\right)^k \left(\mathbf{A}_a - \mathbf{A}_b\right) \cdot \nabla w_{ab} \end{aligned}$$

  - Same conclusions as above
  - Same thing with a 'minus' sign in the gradient, called $\mathbf{G}_a^-$

Effect of background pressure

- The **backround pressure** modifies the state equation:

$$p_a = \frac{\rho_0 c_0^2}{\gamma} \left(\frac{\rho_a^\gamma}{\rho_0^\gamma} - 1 + D \right)$$

- Note: this is only relevant with the $\mathbf{G}_a^+$ (or $\mathbf{G}_a^k$) SPH gradient operators, not with $\mathbf{G}_a^-$.

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- The theory remains unchanged except F_1:

$$F_1 \left(K^+, p^+ \right) \doteq K^{+2} \widehat{w_h} \left(K^+ \right) \left[p^+ + \left(1 - p^+ \right) \widehat{w_h} \left(K^+ \right) \right]$$

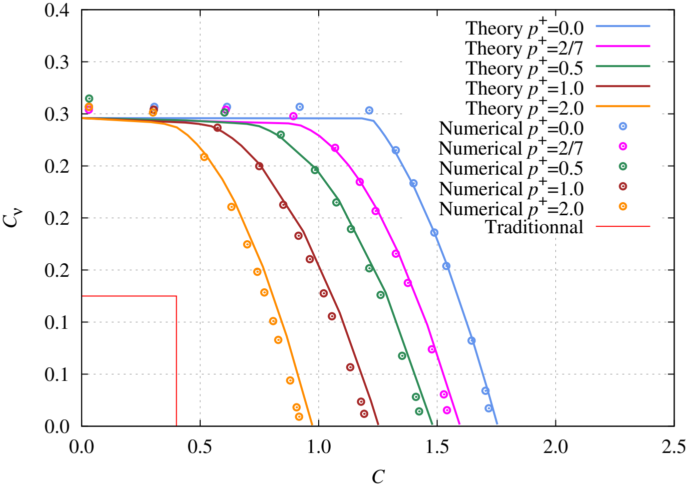
where

$$p^+ \doteq \frac{2D}{\gamma} = \frac{2p_0}{\rho_0 c_0^2}$$

is a dimensionless background pressure.

Effect of background pressure: validation

- 'Infinite flow' test case, $n = 2$, Wendland kernel with $\mathbf{G}_a^+$



Model variations (2)

- Using Monaghan and Gingold (1983)'s SPH Laplacian:

$$\mathbf{L}^{MG}\left\{\mathbf{A}\right\} \doteqdot 2\left(n+2\right)\int_{\Omega}\left[\mathbf{A}\left(\mathbf{r}\right)-\mathbf{A}\left(\mathbf{r}'\right)\right]\cdot\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^{2}}\nabla_{\mathbf{r}}w_{h}\left(\left|\mathbf{r}-\mathbf{r}'\right|\right)d\mathbf{r}'$$

$$\mathbf{L}_{a}^{MG}\left\{\mathbf{A}_{b}\right\} \doteqdot 2\left(n+2\right)\sum_{b}V_{b}\left(\mathbf{A}_{a}-\mathbf{A}_{b}\right)\cdot\frac{\mathbf{r}_{ab}}{r_{ab}^{2}}\nabla w_{ab}$$

Monaghan, J.J. and Gingold, R.A. (1983), J. Comput. Phys. **52**(2):374–389

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$$\mathbf{L}_a^{MG}\left\{\mathbf{A}_b\right\} \doteqdot 2\left(n+2\right)\sum_{b}V_b\left(\mathbf{A}_a-\mathbf{A}_b\right)\cdot\frac{\mathbf{r}_{ab}}{r_{ab}^{2}}\nabla w_{ab}$$

- The function F_2 should then be modified as follows:

$$F_2^{MG}\left(K^+\right) \doteqdot \frac{n+2}{n}\left[F_2\left(K^+\right)+\left(n-1\right)b\left(K^+\right)\right]$$

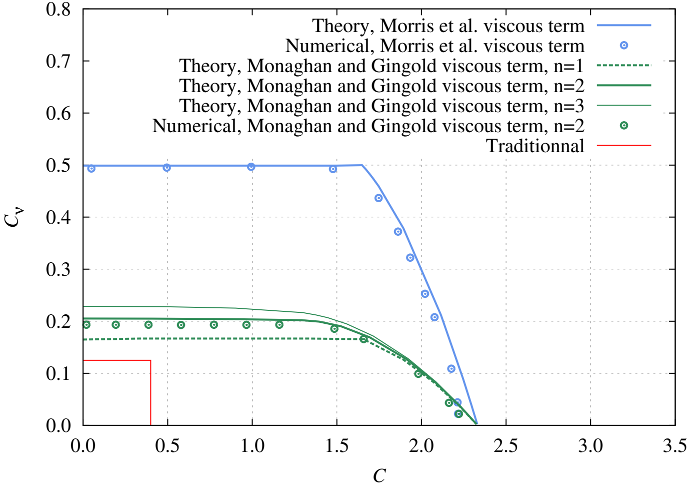
where

$$b\left(K^+\right) \doteqdot \frac{2}{K^{+n}}\int_0^{K^+}\kappa^{n+1}\widehat{w_h}\left(\kappa\right)d\kappa$$

Monaghan, J.J. and Gingold, R.A. (1983), J. Comput. Phys. **52**(2):374–389

Model variations (3)

- 'Infinite flow' case, Gaussian kernel with both Laplacians



Effect of the time marching scheme

- Using **old** velocities to update positions and densities (**fully explicit** scheme), there is no more χ in the r-h-s:

$$
\begin{aligned}
\frac{\chi - 1}{\delta t}\mathbf{U}_0 &= \frac{ic_0}{h}\widehat{w_h}\left(K^+\right)R_0\mathbf{K}^+ - \frac{\nu}{h^2}F_2\left(K^+\right)\mathbf{U}_0 \\
\frac{\chi - 1}{\delta t}\mathbf{R}_0 &= \frac{c_0}{h}\mathbf{U}_0 \\
\frac{\chi - 1}{\delta t}R_0 &= \frac{ic_0}{h}\widehat{w_h}\left(K^+\right)\mathbf{K}^+ \cdot \mathbf{U}_0
\end{aligned}
$$

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\frac{\chi - 1}{\delta t}R_0 &= \frac{ic_0}{h}\widehat{w_h}\left(K^+\right)\mathbf{K}^+ \cdot \mathbf{U}_0
\end{aligned}$$

- The characteristic polynomial now reads:

$$\chi^2 + \left(A_2 - 2\right)\chi + 1 + A_1 - A_2 = 0$$

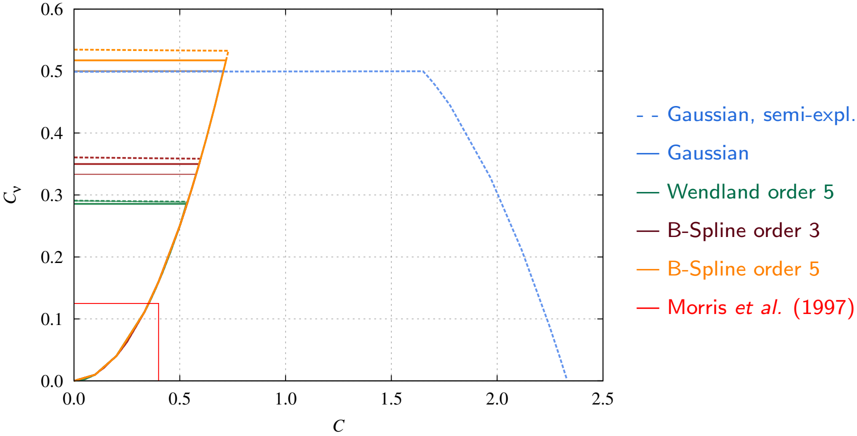
- The stability criterion $\forall \mathbf{K}^+, |\chi| \leq 1$ is modified:

$$C^2 \leqslant C_\nu \leqslant \frac{2}{\lim\limits_{K^+ \longrightarrow +\infty} F_2\left(K^+\right)}$$

Stability domains: fully explicit scheme

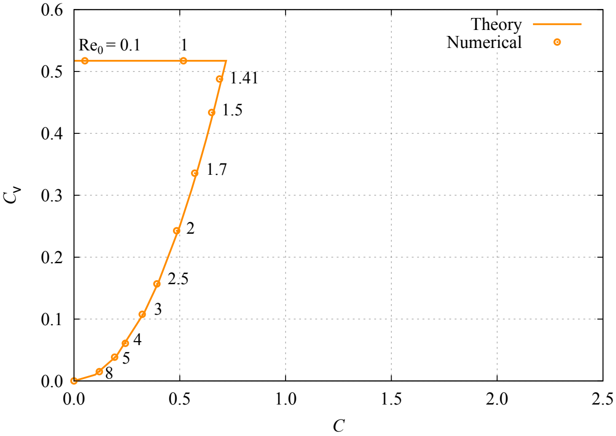
$$C^2 \leqslant C_\nu \leqslant \frac{2}{\lim\limits_{K^+ \longrightarrow +\infty} F_2\left(K^+\right)}$$

Similar to De Leffe, 2011



Fully explicit scheme: validation

- 'Infinite flow' case in dimension $n = 2$, B-spline order 5 with fully explicit scheme

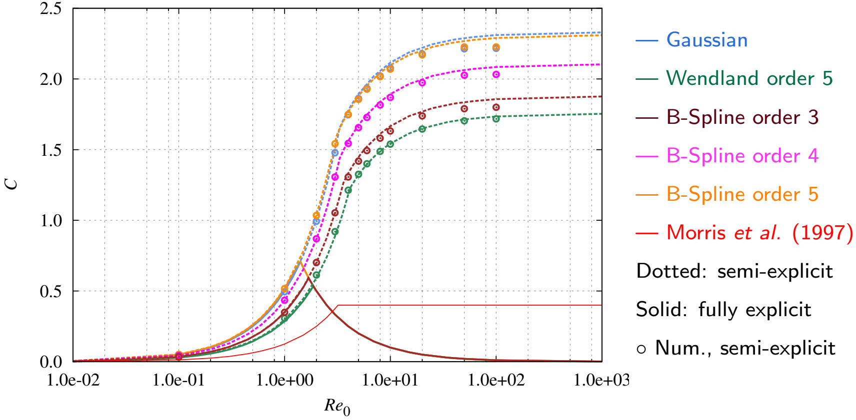


— Theory

○ Numerical

Maximum CFL number

- Plotting the maximum value of C *vs* Re_0 ($n = 2$)



Sensitivity to various parameters

- Some model options may be modified **without modifications** on the numerical results:
  - $h/\delta r = 1.2$ instead of 1.5
  - Random initial density noise instead of vertical discontinuity
  - Initial particle distribution: Cartesian or triangular packaging

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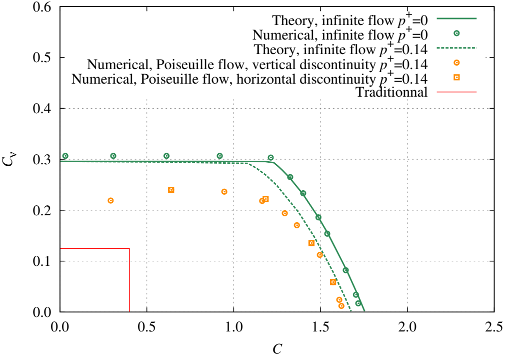
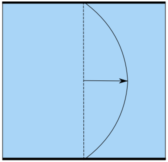
  - $h/\delta r = 1.2$ instead of 1.5
  - Random initial density noise instead of vertical discontinuity
  - Initial particle distribution: Cartesian or triangular packaging
- Effect of a **velocity gradient**:
  - Linearizing around a reference state with a uniform velocity gradient $\mathbf{u}_{ref} = \frac{z}{T}\mathbf{e}_x$ (T^{-1} = rate of strain) gives a more complex eigenvalue problem
  - A polynomial of order 5 is obtained for χ, involving $C_T \doteq \frac{\delta t}{T}$
  - However, in practice C_T is so small that velocity gradients have **almost no effect** on the stability domain
  - This is confirmed by numerical experiments
  - Same conclusions for pressure gradients

Wall effect

- Including wall effects in the theory is not that easy:
  - **Boundary integrals** occur
  - Numerical waves are **reflected** onto the wall so that the resulting wave should fulfill the wall acoustic **boundary condition**
- Tests on a **Poiseuille flow** (with background pressure):



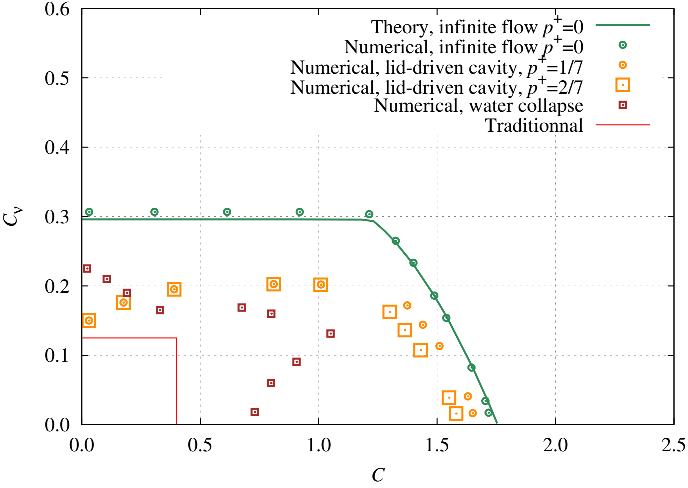


Application to 'real' flows (1)

- Experimental stability domain for:
  - The **lid-driven cavity** (steady, no free surface)
  - A **water collapse** on a wedge (unsteady, free surface)
- Simulations done by Agnès Leroy

Application to 'real' flows (2)

- 'Real' 2-D flows, Wendland kernel



Conclusions (1)

- The present approach used two approximations:
  - **Continuous** SPH differential operators
  - **Linearized** forms of the governing equations
- The theory provides **stability domains** for the time step including the effects of **various model options**:
  - Arbitrary space dimension n
  - Kernel choice (through the $\widehat{w_h}$, F_1 and F_2 functions)
  - Continuity equation or density interpolation
  - Various gradient, divergence and laplacian forms
  - Background pressure
  - Various time marching schemes
- **Experimental tests** are in excellent agreement with the theory
- The numerical **Reynolds number** could nor exceed ~ 100
- **Wall effects** remain difficult to treat

Conclusions (2)

- The following **recommendations** follow:
  - The time step can be larger than in Morris *et al.*'s work
  - The stability domain is almost independent on the kernel for a given resolution σ
  - No matter the way the density is computed
  - No matter the forms of gradient or divergence operators
  - Morris *et al.*'s Laplacian is better than Monaghan and Gingold's
  - Do not use fully explicit time integration schemes

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  - No matter the way the density is computed
  - No matter the forms of gradient or divergence operators
  - Morris *et al.*'s Laplacian is better than Monaghan and Gingold's
  - Do not use fully explicit time integration schemes
- **Other features** can be treated the same way:
  - Surface tension: $C \leq f(C_\nu, C_\beta)$, $C_\beta \doteqdot \frac{\beta \delta t^2}{\rho_0 h^3} = \frac{C^2}{We_0}$
  - Density smoothing, Incompressible SPH
  - Solids, MHD and other kinds of Physics
  - Higher order time marching schemes (Leapfrog, etc.), but they lead to higher degree polynomials for χ
  - Similar methods: MPS, FVPM, DPD, etc.

Further references

- Violeau, D., *Fluid Mechanics and the SPH Method. Theory and Applications*, Oxford Univ. Press, 2012.
- Violeau, D. and Leroy, A., *Maximum time step for keeping numerical stability of viscous weakly compressible SPH*, submitted to the J. Comput. Phys.

Merci de votre attention.

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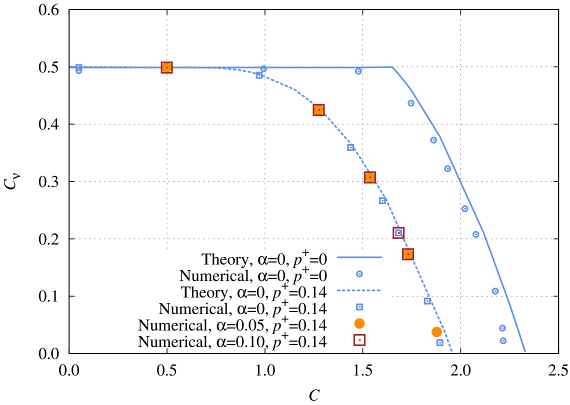
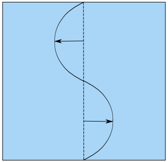
Velocity gradient: validation

- 'Infinite flow' case, $n = 2$, with **sinusoidal velocity** field (note: background pressure was necessary in this case)





$$u_x\left(z\right) = \alpha c_0 \sin \frac{2\pi z}{L}$$

$$T \sim \frac{L}{2\pi\alpha c_0}$$

… with a continuous time

- If we consider the **time** as **continuous** (no time scheme):

$$\begin{aligned}
i\omega\mathbf{U}_0 &= \frac{ic_0}{h}\widehat{w_h}\left(K^+\right)R_0\mathbf{K}^+ - \frac{\nu}{h^2}F_2\left(K^+\right)\mathbf{U}_0 \\
i\omega\mathbf{R}_0 &= \frac{c_0}{h}\mathbf{U}_0 \\
i\omega R_0 &= \frac{ic_0}{h}\widehat{w_h}\left(K^+\right)\mathbf{K}^+\cdot\mathbf{U}_0
\end{aligned}$$

... with a continuous time

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- The dispersion relation reads

$$\omega = \frac{\nu}{2h^2}F_2\left(K^+\right)\left[i \pm \sqrt{4{Re_0}^2\frac{F_1\left(K^+\right)}{F_2\left(K^+\right)^2} - 1}\right]$$

- $\mathrm{Im}\,\omega \geq 0$ for all K^+: the system is **always stable**!